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Universal minimal flows from a homotopical perspective

Matúš Dirbák[1]

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Abstract
We study universal minimal flows of connected and locally contractible topological groups by means of tools from homotopy theory and dynamical cohomology. We provide methods for computing their first cohomotopy groups and illustrate them by computations related to connected Lie groups and identity components of homeomorphism groups of compact connected surfaces with or without boundary. Since metrizable spaces have countable first cohomotopy groups, these methods serve as a useful new instrument for proving nonmetrizability of various universal minimal flows. In this way we reprove several results in this direction known before and also prove some new ones. Finally, we indicate how our methods can be used to construct finite-dimensional minimal spaces for infinite-dimensional topological groups, having all orbits of first category.

Keywords Minimal flow · Universal minimal flow · Dynamical cohomology · First cohomotopy group

Mathematics Subject Classification Primary 37B05

1 Introduction

1.1 Morphisms of minimal flows

The main object of study in abstract topological dynamics is a flow $\mathcal{F}\colon \Gamma \curvearrowright X$, given by a continuous action of a topological group Γ on a topological, usually compact space X. The acting transformations of $\mathcal{F}$, denoted by T_γ ($\gamma \in \Gamma$), are homeomorphisms of X and we think of $\mathcal{F}$ as a topological representation of Γ. Having originated from the classical cases $\Gamma = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, topological dynamics is concerned with the asymptotic behaviour of orbits; if all of them are dense then $\mathcal{F}$ is called minimal. Given a topological group Γ, the Γ-flows form a category; a morphism $h\colon \mathcal{G} \to \mathcal{F}$ is a continuous equivariant map. If h is also surjective then $\mathcal{G}$ is called an extension of $\mathcal{F}$ and $\mathcal{F}$ a factor of $\mathcal{G}$. Morphisms form a fundamental ingredient in the structure theory of minimal flows and their understanding is thus one of the goals of topological dynamics.

✉ Matúš Dirbák
Matus.Dirbak@umb.sk

[1] Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, Slovakia

In particular, the projection π of $\mathcal{F}$ onto its maximal equicontinuous factor is a basic object of interest in dynamics. Despite its relevance for various considerations, little seems to be known about its topological properties. Monotonicity of π, proved by Hauser and Jäger for flows on locally connected spaces [32, Theorem A], enables to characterize maximal equicontinuous factors of minimal $\mathbb{Z}$-flows on the 2-torus [32, Theorem B]. Remarkably, the notion of maximal equicontinuous factor also arises in the study of tilings and minimal flows of $\mathbb{R}^n$ ($n \in \mathbb{N}$) on tiling spaces. By a result of Barge, Kellendonk and Schmieding, the corresponding factor map π induces a monomorphism between the first Čech cohomology groups [3, Corollary 3]. Moreover, the cokernel of the morphism induced by π is torsion-free [3, Theorem 5].

It turns out that the injectivity of the induced morphism is true for every morphism of minimal flows, provided that their acting group Γ is connected and locally contractible; proved by methods developed in Sect. 4, this result is better expressed in the language of cohomotopy theory rather than cohomology.

Theorem 1.1 *Let Γ be connected and locally contractible, $\mathcal{F}$ and $\mathcal{G}$ be minimal Γ-flows on compact spaces X and Y, respectively, and $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism. Then the induced morphism $h^\flat\colon \pi^1(X) \to \pi^1(Y)$ is a monomorphism, hence so is $h^\flat\colon \pi^1(\mathcal{F}) \to \pi^1(\mathcal{G})$.*

Here $\pi^1(X)$ denotes the first cohomotopy group of X which is known to be isomorphic to the first Čech cohomology group; for definitions and basic properties of $\pi^1(X)$ and its subgroup $\pi^1(\mathcal{F})$, see Sects. 2.9 and 3.2, respectively.

It follows from Theorem 1.1 that for a morphism between minimal flows on manifolds the induced morphism between the first weak homology groups has maximal rank (Corollary 4.10). Similarly to the theorem of Hauser and Jäger mentioned above, Theorem 1.1 and Corollary 4.10 impose a strong restriction on possible phase spaces of a maximal equicontinuous factor (Proposition 4.12). Finally, the torsion-freeness of the cokernel of the induced morphism from [3] is shared by the maximal equicontinuous factor map only and it does not extend to general morphisms of minimal flows (Example 4.11).

1.2 Universal minimal flows of homeomorphism groups

Given a topological group Γ, the category of (compact) minimal Γ-flows has a universal object $\mathcal{U}(\Gamma)\colon \Gamma \curvearrowright M(\Gamma)$, called the universal minimal flow of Γ; it is defined as a common extension of all minimal Γ-flows. First described by Ellis for discrete groups [14], it was later realized that it exists and is unique up to isomorphism for every topological group Γ (see, e.g., [26]). The complexity of $\mathcal{U}(\Gamma)$ varies strongly with Γ. While $\mathcal{U}(\Gamma)$ acts freely when Γ is locally compact [14, Theorem 3], [54, Theorem 2.2.1], for general Polish groups $M(\Gamma)$ may even reduce to a singleton – this is true precisely when every continuous action of Γ on a compact space has a fixed point, in which case Γ is called extremely amenable or is said to have the fixed point on compacta property.

The notion of extreme amenability for semigroups dates back to works of Granirer [24] and Mitchell [43], but the first example in the class of topological groups is due to Herer and Christensen [33]. Considered at first exotic, it was subsequently realized that this property is not rare among Polish groups. Being shared by the unitary group of ℓ_2 [25], the automorphism group of the rational linear order [47], isometry groups of Lebesgue's L^p-spaces [15] and many others [6, 16, 39, 48], it may even be present in monothetic groups [17].

When nontrivial, $\mathcal{U}(\Gamma)$ is usually hard to describe explicitly. The first result providing a concrete description of a nontrivial universal minimal flow is due to Pestov who identified the evaluation flow of $\Gamma = \mathcal{H}_+(\mathbb{T})$, the group of orientation preserving homeomorphisms of the circle $\mathbb{T}$ (equipped with the compact-open topology), as universal among all minimal Γ-flows, thus yielding $M(\Gamma) = \mathbb{T}$ [47, Theorem 6.6]. For the real line $\mathbb{R}$ and the unit compact interval $\mathbb{I}$, the corresponding groups $\mathcal{H}_+(\mathbb{R})$ and $\mathcal{H}_+(\mathbb{I})$ are extremely amenable [47, Theorem 6.2].

Shortly afterwards Uspenskij proved that the evaluation flow can not be universal for homeomorphism groups $\mathcal{H}(X)$ of higher-dimensional compact connected manifolds X without boundary, since no universal minimal flow acts transitively on ordered tripples of distinct points of $M(\Gamma)$ [53]. In fact, by a recent result of Gutman, Tsankov and Zucker, phase spaces $M(\Gamma)$ of locally transitive subgroups Γ of $\mathcal{H}(X)$ are not metrizable [27, Theorem 1.1]. (Recall that local transitivity of Γ requires that for each $x \in X$ and every neighbourhood U of x there exist a neighbourhood $V \subseteq U$ of x such that for each $y \in V$ there is $T \in \Gamma$ with $T(x) = y$ and $T(z) = z$ for every $z \in X \setminus U$.) It is not difficult to see that the identity component $\mathcal{H}_0(X)$ of $\mathcal{H}(X)$ is locally transitive, so the mentioned result from [27] applies to it. For manifolds with boundary, having dimension at least 3, a similar result immediately follows (Proposition 5.24). Although the argument used to prove Proposition 5.24 does not apply to compact surfaces X, the nonmetrizability of $M(\Gamma)$ for $\Gamma = \mathcal{H}_0(X)$ in this case can be proved by methods of this paper when X is not homeomorphic to the disc and the Möbius band (Proposition 5.25 and Example 5.26).

Apart from 1-dimensional manifolds, the universal minimal flow is known for homeomorphism groups of other low-dimensional spaces, including the Cantor set [22] and, more generally, some totally disconnected compacta [18], or the Lelek fan [5], but for spaces of higher dimensions, manifolds in particular, $\mathcal{U}(\Gamma)$ and $M(\Gamma)$ remain unidentified.

1.3 Algebraic topology of universal minimal flows

Topological groups allowing explicit description of a nontrivial $\mathcal{U}(\Gamma)$ are rare. Besides homeomorphism groups listed in the preceding section, they include the symmetric group of integers [21] and automorphism groups of countable structures [39]. It is remarkable that for discrete groups, which are of interest in abstract topological dynamics since its origins, $\mathcal{U}(\Gamma)$ is currently unavailable, though its phase space $M(\Gamma)$ is well understood [1, 20, 52]. As for general locally compact groups, even the space $M(\Gamma)$ is known only in very special cases [4, 8, 38, 51].

As far as we are able to say, the studies of universal minimal flows have not yet included algebraic topology of $M(\Gamma)$. Though the basic algebraic part of topology is usually concerned with computing algebraic invariants of concrete spaces, in the case when little is known about the space these invariants may provide a valuable information about its topology. With this idea as a motivation, we aim at developing tools for computing the first cohomotopy group $\pi^1(M(\Gamma))$ of $M(\Gamma)$, focusing on topological groups Γ which are connected, locally contractible and separable. On the one hand, these groups allow using methods from the theory of covering spaces and fundamental group; on the other hand, they include important families of groups such as connected Lie groups and identity components of homeomorphism groups of compact connected manifolds.

It turns out then that $\pi^1(M(\Gamma))$ splits into the direct sum of an isomorphic copy of the real dynamical cohomology group $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ of $\mathcal{U}(\Gamma)$ and a subgroup of $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$, where $\pi_1(\Gamma)$ is the fundamental group of Γ (Proposition 5.2); when $M(\Gamma)$ is metrizable then $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) = 0$, hence $\pi^1(M(\Gamma))$ itself is isomorphic to a subgroup of $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$ (Corollary 5.4). In the case when Γ has a finitely generated fundamental group the two direct summands of $\pi^1(M(\Gamma))$ are elementary.

Theorem 1.2 *Let Γ be separable, connected, locally contractible and with a finitely generated fundamental group. Then there is a direct sum*

$$\pi^1(M(\Gamma)) = \pi^1(\mathcal{U}(\Gamma)) \oplus A_\Gamma, \tag{1.1}$$

where

$$\pi^1(\mathcal{U}(\Gamma)) = \mathrm{Div}(\pi^1(M(\Gamma))) \cong \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \tag{1.2}$$

is either 0 *or isomorphic to* $\mathbb{R}$, *and* A_Γ *is a free abelian group with a finite rank*

$$\mathrm{r}(A_\Gamma) = \mathrm{r}(\mathbf{A}_{\mathcal{U}(\Gamma)}) = \max_{\mathcal{F}} \mathrm{r}(\mathbf{A}_{\mathcal{F}}) \le \mathrm{r}(\pi_1(\Gamma)), \tag{1.3}$$

with $\mathcal{F}$ *running through all minimal* Γ*-flows. Moreover, the following conditions are equivalent:*

(1) $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$,
(2) $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \neq 0$,
(3) $\mathbf{H}_{\mathcal{F}}(\mathbb{R}) \neq 0$ *for some minimal flow* $\mathcal{F}\colon \Gamma \curvearrowright X$.

As above, $\pi^1(\mathcal{U}(\Gamma))$ is the subgroup of $\pi^1(M(\Gamma))$ from Definition 3.10, $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ and $\mathbf{H}_{\mathcal{F}}(\mathbb{R})$ are the real cohomology groups of $\mathcal{U}(\Gamma)$ and $\mathcal{F}$ (see Sect. 2.3), $\mathbf{A}_{\mathcal{U}(\Gamma)}$ and $\mathbf{A}_{\mathcal{F}}$ are subgroups of the cocycle groups $\mathbf{Z}_{\mathcal{U}(\Gamma)}(\mathbb{T})$ and $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ defined by (2.17), $\mathrm{Div}(A)$ stands for the group of all divisible elements of an abelian group A and $\mathrm{r}(A)$ denotes the rank of A.

In particular, if Γ is simply connected then the second direct summand in (1.1) vanishes, leaving a dichotomy $\pi^1(M(\Gamma)) \cong \mathbb{R}$ or $\pi^1(M(\Gamma)) = 0$ (Corollary 5.8). It is of importance for practical calculations that the two direct summands $\pi^1(\mathcal{U}(\Gamma))$ and A_Γ of $\pi^1(M(\Gamma))$ can be determined by considering, instead of $\mathcal{U}(\Gamma)$, possibly more concrete and simpler minimal actions $\mathcal{F}$ of Γ. In the case when Γ is a connected Lie group it may even suffice taking into account appropriate closed subgroups of Γ (Corollary 5.17); this is proved with the aid of notions of topologically free flows and flows with free cycles introduced in [12, Sect. 2.5.2] and studied in [13] (Proposition 5.16). In this way we compute $\pi^1(M(\Gamma))$ for connected abelian Lie groups, Heisenberg groups, $\mathrm{SL}_2(\mathbb{R})$ and its universal cover, or $\mathrm{GL}_2^+(\mathbb{R})$. Finally, we determine $\pi^1(M(\Gamma))$ for identity components of homeomorphism groups of some compact connected surfaces with or without boundary (Propositions 5.22 and 5.25), for universal covers of such groups for certain compact connected manifolds without boundary (Proposition 5.12) and for identity components of loopgroups of some compact connected Lie groups (Proposition 5.14).

Since the group $\pi^1(M(\Gamma))$ is countable when $M(\Gamma)$ is metrizable, it seems that its computation might be a useful new method for verifying nonmetrizability of $M(\Gamma)$.

1.4 Orbits of universal minimal flows

If Γ is a σ-compact group and $\mathcal{F}\colon \Gamma \curvearrowright X$ is minimal but not transitive then it is elementary to verify that all orbits of $\mathcal{F}$ are of first category, hence the same is true for $\mathcal{U}(\Gamma)$ (see, e.g., Lemma 5.27). Topological size of orbits of universal minimal flows associated to general Polish groups is, on the contrary, a nontrivial problem. While $\mathcal{U}(\Gamma)$ has a generic orbit when $M(\Gamma)$ is metrizable [9, 56], the converse implication is in general not true [40]. By a theorem of Gutman, Tsankov and Zucker, for every locally transitive subgroup Γ of the homeomorphism group $\mathcal{H}(X)$ of a compact connected manifold X without boundary and of dimension at least 3, all orbits of $\mathcal{U}(\Gamma)$ are of first category [27, Theorem 1.2]. This result was recently extended by Basso, Codenotti and Vaccaro to include identity components of homeomorphism groups of closed surfaces X, apart from the sphere and the projective plane [7, Theorem 1.3]. A common feature of these two results is their relying on the induced action of Γ on the space of maximal chains of subcontinua of X. As it turns out, for surfaces X with negative Euler characteristic it suffices to consider actions of $\Gamma = \mathcal{H}_0(X)$ on much simpler compacta.

Theorem 1.3 *Let X be a compact connected surface without boundary and with $\chi(X) < 0$, and let $\Gamma = \mathcal{H}_0(X)$ be the identity component of $\mathcal{H}(X)$. Then there is a minimal Γ-flow on $X \times \mathbb{T}$, whose all orbits are of first category in $X \times \mathbb{T}$. Consequently, each orbit of $\mathcal{U}(\Gamma)$ is of first category in $M(\Gamma)$.*

The minimal Γ-flow on $X \times \mathbb{T}$ from Theorem 1.3 is constructed in the form of the group extension $\mathcal{F}_{\mathcal{C}}$ of the evaluation flow $\mathcal{F}$ of Γ on X by an appropriate cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ as defined in (2.2) (Proposition 5.30). The four remaining surfaces X, namely the sphere, the projective plane, the torus and the Klein bottle, do not seem to be directly attainable by methods of this paper, though the latter two are of course covered by [7, Theorem 1.3].

1.5 Standing assumptions

We work with Hausdorff topological spaces only. Throughout the rest of this paper $\mathcal{F}$ denotes a minimal flow. Its acting group will be denoted by Γ, the phase space by X, acting transformations by T_γ ($\gamma \in \Gamma$) and the orbit of $x \in X$ by $\mathcal{O}_{\mathcal{F}}(x)$. We assume that X is *compact and connected*. If metrizability is imposed upon X, we express it explicitly. It is also assumed that X is carrying a base point which we denote by z. Although our main interest is in flows with connected acting groups, we shall impose connectedness (and other restrictions on the topology of Γ) only when necessary. In particular, we do not restrict our attention to the class of locally compact groups. Since we are about to work with compactifications of noncompact abelian group extensions of $\mathcal{F}$, we consider also flows with noncompact phase spaces. However, we speak of minimality only in the context of flows defined on compacta.

2 Preliminaries

This section collects some preliminary material from algebra, topology and topological dynamics that will be extensively used throughout this work. While most of the concepts considered here are well known, we also introduce some new notions, such as the induced exponential $\mathrm{Exp}_{\mathcal{F}}$ and the group $\mathbf{A}_{\mathcal{F}}$ in Sects. 2.16 and 2.17, which will play an important role in our analysis of universal minimal flows. Finally, we wish to mention that certain parts of this section (and of Sect. 3 below) are written in a way that serves the purpose of this paper as well as the purpose of a related forthcoming paper which will focus on the structure of minimal compactifications of abelian extensions of minimal flows.

2.1 Abelian groups

We denote the group operations of an abelian group A either additively or multiplicatively, depending on what we consider more appropriate. The *torsion-free rank* of A, denoted by $\mathrm{r}(A)$, is the cardinality of a maximal among subsets of A formed by elements independent over $\mathbb{Z}$. Given a short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0,$$

we have $\mathrm{r}(B) = \mathrm{r}(A) + \mathrm{r}(C)$.

If G is an abelian topological group then G_d is the topological group obtained from G by changing its topology to the discrete one. Completeness of an abelian topological group is considered with respect to the uniformity generated by identity neighbourhoods and requires that every Cauchy net be convergent. Every locally compact abelian group is complete. Given $n \in \mathbb{N}$, we denote the n-torus by $\mathbb{T}^n$ and abbreviate $\mathbb{T}^1$ to $\mathbb{T}$.

2.2 The compact-open topology

Let Y be a topological space and G be an abelian topological group. Denote by $C(Y, G)$ the set of all continuous maps $Y \to G$. With operations defined pointwise, $C(Y, G)$ is an abelian group. If Y is a topological group then $\mathrm{Hom}(Y, G)$ denotes the subgroup of $C(Y, G)$ formed by the topological (that is, continuous) morphisms $Y \to G$.

Now let Y be a locally compact space. Recall that the *compact-open topology* on $C(Y, G)$ is generated by the sets

$$[K, U] := \{f \in C(Y, G) \colon f(K) \subseteq U\}, \tag{2.1}$$

where $K \subseteq Y$ is compact and $U \subseteq G$ is open. With this topology $C(Y, G)$ is a topological group, a local basis at the identity being formed by the sets $[K, U]$, where U runs through identity neighbourhoods in G. If G is complete then so is $C(Y, G)$. If Y is second countable and G is completely metrizable (respectively, separable metrizable) then $C(Y, G)$ is also completely metrizable (respectively, separable metrizable).

Given a pointed topological space Y with the base point w, let $C_w(Y, G)$ be the subgroup of $C(Y, G)$ formed by the maps f with $f(w) = 0$. If Y is locally compact then $C_w(Y, G)$ is closed in $C(Y, G)$.

2.3 Dynamical cohomology

Let $\mathcal{F} \colon \Gamma \curvearrowright X$ be a minimal flow, G be an abelian topological group and $\mathcal{C} \colon \Gamma \times X \to G$ a continuous map. The assignment

$$\mathcal{F}_{\mathcal{C}} \colon \Gamma \times (X \times G) \to X \times G, \quad (\gamma, (x, g)) \mapsto (T_\gamma x, \mathcal{C}(\gamma, x)g), \tag{2.2}$$

defines a Γ-flow on $X \times G$ if, and only if, $\mathcal{C}$ satisfies the *cocycle identity*

$$\mathcal{C}(\alpha, T_\beta x)\mathcal{C}(\beta, x) = \mathcal{C}(\alpha\beta, x). \tag{2.3}$$

If that is the case then $\mathcal{C}$ is called a *cocycle* over $\mathcal{F}$. It is called a *coboundary* when there exists a continuous map $\xi \colon X \to G$ with

$$\mathcal{C}(\gamma, x) = \xi(T_\gamma x)\xi(x)^{-1}; \tag{2.4}$$

we write $\mathcal{C} = \mathrm{co}(\xi)$ and refer to ξ as a *transfer function* of $\mathcal{C}$. By minimality of $\mathcal{F}$, any two transfer functions of a given coboundary $\mathcal{C}$ differ by a constant; in particular, given $z \in X$, there is a unique transfer function of $\mathcal{C}$ mapping z to 1. Due to this uniqueness property we shall preferably use transfer functions that preserve base points – this may be assumed without further notice.

Cocycles form an abelian group under pointwise operations, denoted by $\mathbf{Z}_{\mathcal{F}}(G)$, of which coboundaries constitute a subgroup $\mathbf{B}_{\mathcal{F}}(G)$. The quotient group

$$\mathbf{H}_{\mathcal{F}}(G) := \mathbf{Z}_{\mathcal{F}}(G)/\mathbf{B}_{\mathcal{F}}(G) \tag{2.5}$$

is referred to as the *cohomology group* of $\mathcal{F}$ relative to G. For an explanation of the cohomological terminology used in this setting, see [50, Section 1.21]. When the acting group Γ of $\mathcal{F}$ is locally compact then we let $\mathbf{Z}_{\mathcal{F}}(G)$ carry the compact-open topology, thus turning it into an abelian topological group.

For the sake of fluency of presentation, we shall attribute to $\mathcal{C}$ the dynamical properties of $\mathcal{F}_{\mathcal{C}}$; in particular, we call $\mathcal{C}$ minimal when $\mathcal{F}_{\mathcal{C}}$ is minimal.

2.4 Basic topology of $\mathbf{Z}_{\mathcal{F}}(G)$

If Γ is locally compact and G is an abelian topological group then $\mathbf{Z}_{\mathcal{F}}(G)$ is a closed subgroup of $C(\Gamma \times X, G)$. Following our discussion in Sect. 2.2, we infer that $\mathbf{Z}_{\mathcal{F}}(G)$ is

- complete if G is complete (in particular, if G is locally compact),
- completely metrizable if Γ is second countable, X is metrizable and G is completely metrizable,
- separable metrizable if Γ is second countable, X is metrizable and G is separable metrizable.

2.5 Pontryagin duality

Let G be a locally compact abelian group and $H \subseteq G$ be a closed subgroup. We denote by G^* the *Pontryagin dual* of G and by $H^\perp$ the *annihilator* of H in G^*. Recall that G is compact if, and only if, G^* is discrete. To every topological morphism of locally compact abelian groups $q\colon G \to K$ we associate the *dual topological morphism* $q^*\colon K^* \to G^*$. Recall that q^* is a monomorphism if, and only if, q has a dense image. We have the Pontryagin topological isomorphism $G \to G^{**}$, which maps every closed subgroup H of G onto $H^{\perp\perp}$. In this way $q\colon G \to K$ becomes identified with $q^{**}\colon G^{**} \to K^{**}$.

A *compactification* of $\mathbb{R}$ consists of a compact connected abelian group G and a topological morphism $q\colon \mathbb{R} \to G$ with a dense image. Up to topological isomorphism, there is a unique compactification of $\mathbb{R}$ which is maximal in the sense that it projects onto every compactification of $\mathbb{R}$. We denote it by $p\colon \mathbb{R} \to b\mathbb{R}$ and call it the *Bohr compactification* of $\mathbb{R}$. It may be constructed by setting $b\mathbb{R} = (\mathbb{R}^*_d)^*$, with $\mathbb{R}^*_d$ being the discretized group $\mathbb{R}^*$, and defining p by letting p^* be the composition of the inverted Pontryagin isomorphism $(\mathbb{R}^*_d)^{**} \to \mathbb{R}^*_d$ with the identity $\mathbb{R}^*_d \to \mathbb{R}^*$. Recall that every compact connected separable abelian group G is a compactification of $\mathbb{R}$, hence is topologically isomorphic to a quotient group of $b\mathbb{R}$. We shall also consider the compactification of $\mathbb{R}$ via the *exponential* $\exp\colon \mathbb{R} \to \mathbb{T}$, defined in the complex notation by $\exp(x) := e^{i2\pi x}$. Given $t \in \mathbb{R}$, we denote by χ_t the character of $\mathbb{R}$ defined by

$$\chi_t(x) := \exp(tx). \tag{2.6}$$

The assignment $t \mapsto \chi_t$ defines a topological isomorphism $\mathbb{R} \to \mathbb{R}^*$.

2.6 Fundamental group

If X is a pointed topological space with the base point z, we denote its *fundamental group* by $\pi_1(X, z)$. When no confusion may arise concerning the base point, we write $\pi_1(X)$ instead of $\pi_1(X, z)$. Recall that the product in $\pi_1(X)$ is represented by concatenation of loops and the inverse of a loop is its reverse loop. We denote by $f * g$ the concatenation of paths f, g and by $\overline{f}$ the path reverse to f. If X, Y are pointed topological spaces and $h\colon X \to Y$ is a continuous base point preserving map then the *induced morphism* $\pi_1(X) \to \pi_1(Y)$ is denoted by h_*. Recall that the assignment $h \mapsto h_*$ is covariant functorial.

For a topological group Γ we use its identity as the base point. If Γ is a connected Lie group and K is its maximal compact subgroup then K is connected and the inclusion morphism $K \to \Gamma$ is a homotopy equivalence by a theorem of Mal'cev [42] and Iwasawa [36]. It follows that the induced morphism $\pi_1(K) \to \pi_1(\Gamma)$ is an isomorphism [46, Theorem 58.7, p. 364]. Since $\pi_1(X)$ is finitely generated for every compact connected manifold X [31, Corollary A.8 and A.9, p. 527], $\pi_1(\Gamma)$ is finitely generated for every connected Lie group Γ.

2.7 The first weak homology group

Let X be a connected (topological) manifold. The *first homology group* $H_1(X)$ of X will be interpreted solely as the abelianization of $\pi_1(X)$. The quotient group

$$H_1^w(X) := H_1(X)/\operatorname{tor}(H_1(X)) \tag{2.7}$$

is abelian and torsion-free, and will be referred to as the *first weak homology group* of X. Thus $H_1^w(X)$ is a quotient group of $\pi_1(X)$. If $h\colon X \to Y$ is a continuous map between connected manifolds then the induced morphism h_* descends to a morphism $h^\sharp\colon H_1^w(X) \to H_1^w(Y)$. The assignment $h \mapsto h^\sharp$ is covariant functorial.

If $\pi_1(X)$ is finitely generated then $H_1^w(X)$ is free abelian of a finite rank and it may be identified with the torsion-free part of $H_1(X)$.

2.8 Topological freeness and free cycles

Let $\mathcal{F}$ be a minimal flow, given by an action of a connected Lie group Γ on a compact (connected) manifold X. Consider the *transition map* of $z \in X$,

$$\mathcal{F}_z\colon \Gamma \to X, \quad \gamma \mapsto \mathcal{F}(\gamma, z), \tag{2.8}$$

and the induced morphism $(\mathcal{F}_z)^\sharp\colon H_1^w(\Gamma) \to H_1^w(X)$. The image of $(\mathcal{F}_z)^\sharp$ will be denoted by

$$H_1^w(\mathcal{F}) := \operatorname{im}((\mathcal{F}_z)^\sharp). \tag{2.9}$$

Since $H_1^w(X)$ is a free abelian group of a finite rank, the same is true for $H_1^w(\mathcal{F})$. Following [12, Section 2.5.2], we say that

- $\mathcal{F}$ is *topologically free* if $(\mathcal{F}_z)^\sharp$ is a monomorphism,
- $\mathcal{F}$ *has a free cycle* if $\mathrm{r}(H_1^w(\mathcal{F})) < \mathrm{r}(H_1^w(X))$.

As explained in [12, Remark 2.8, p. 46], these two notions do not depend on the choice of a base point z.

2.9 The first cohomotopy group

Let X be a compact connected space and write $A = C(X, \mathbb{T})$. The compact-open topology on A coincides with the topology of uniform convergence. Consider the subset A_0 of A, consisting of those elements ξ of A that lift across the exponential $\exp\colon \mathbb{R} \to \mathbb{T}$ to a continuous map $\eta\colon X \to \mathbb{R}$. Then A_0 is a path-connected subgroup of A. Since exp has continuous local cross sections, A_0 is also open in A, hence it coincides with the identity path-component and the identity component of A. Thus A_0 is formed by the null-homotopic maps $X \to \mathbb{T}$. The quotient group A/A_0 is discrete abelian and consists of the path-components of A, that is, of the homotopy classes of maps $X \to \mathbb{T}$. We write $\pi^1(X) := A/A_0$ and call it the *first cohomotopy group* of X [35, p. 205] or the *Bruschlinsky group* of X [35, p. 48]. By connectedness of X, $\pi^1(X)$ is torsion-free. If X is also metrizable then A is separable, hence $\pi^1(X)$ is countable.

For the sake of simplicity of notation we shall often use the same symbol ξ for an element of A and for the corresponding element of $\pi^1(X)$. Recall that when X is a compact connected abelian group then each homotopy class of A contains a unique character of X, so $\pi^1(X)$ is isomorphic to X^* [34, Theorem 8.57, p. 420].

Now let Y also be a compact connected space and $h\colon Y \to X$ be a continuous map. Write $B = C(Y, \mathbb{T})$ and consider the identity component B_0 of B. The assignment $\xi \mapsto \xi h$ defines a morphism of groups $A \to B$ mapping A_0 into B_0, hence descending to a morphism $h^\flat\colon \pi^1(X) \to \pi^1(Y)$. Clearly, $h^\flat$ is contravariant functorial in h.

2.10 Connections between algebraic invariants

Let X be a pointed compact connected space with the base point z. Then every homotopy class $\xi \in \pi^1(X)$ contains a base point preserving map. Since any two such maps are homotopic via a base point preserving homotopy, they induce the same morphism $\pi_1(X) \to \pi_1(\mathbb{T})$ [46, Lemma 58.1, p. 360], which we denote by ξ_*. The assignment $\xi \mapsto \xi_*$ defines a morphism of groups

$$\varrho\colon \pi^1(X) \to \operatorname{Hom}(\pi_1(X), \pi_1(\mathbb{T})).$$

Now let X be a compact connected manifold. By [31, Corollary A.12, p. 529] and [31, Proposition 1B.9, p. 90], ϱ is an epimorphism. Moreover, if a base point preserving map $\xi\colon X \to \mathbb{T}$ satisfies $\xi_* = 0$ then it lifts across exp [46, Lemma 79.1, p. 478], so $\xi = 0$ in $\pi^1(X)$. Thus ϱ is an isomorphism. Finally, since $\pi_1(\mathbb{T}) = H_1^w(\mathbb{T})$, we have an isomorphism

$$\sigma : \pi^1(X) \to \operatorname{Hom}(H_1^w(X), H_1^w(\mathbb{T})), \tag{2.10}$$

defined by $\sigma(\xi) := \xi^\sharp$. If we want to emphasize the dependence of σ on X then we write σ_X instead of σ.

The following lemma will help us deduce Corollary 4.10 from Theorem 1.1 in Sect. 4.4.

Lemma 2.1 *Let X, Y be (pointed) compact connected manifolds and $h\colon Y \to X$ be a continuous base point preserving map. Then for every $\varphi \in \operatorname{Hom}(H_1^w(X), H_1^w(\mathbb{T}))$,*

$$\sigma_Y h^\flat \sigma_X^{-1}(\varphi) = \varphi h^\sharp.$$

Proof Fix φ and choose $\xi \in \pi^1(X)$ with $\varphi = \sigma_X(\xi) = \xi^\sharp$. Then

$$\sigma_Y h^\flat \sigma_X^{-1}(\varphi) = \sigma_Y h^\flat(\xi) = \sigma_Y(\xi h) = (\xi h)^\sharp = \xi^\sharp h^\sharp = \varphi h^\sharp.$$

□

2.11 Homeomorphism groups of manifolds

Given a compact space X, denote by $\mathcal{H}(X)$ the group of all homeomorphisms of X. When equipped with the compact-open topology, $\mathcal{H}(X)$ is a topological group. If X is metrizable then $\mathcal{H}(X)$ is separable. The *evaluation map*

$$\mathcal{H}(X) \times X \to X, \quad (T, x) \mapsto T(x), \tag{2.11}$$

is continuous, hence defines a flow; we will refer to it as the *evaluation flow*.

Now let X be a compact connected manifold with or without boundary. Since X is metrizable, $\mathcal{H}(X)$ is a separable topological group. By a classical result of Černavskiĭ [11], $\mathcal{H}(X)$ is locally contractible, hence locally path-connected. Thus the identity component $\mathcal{H}_0(X)$ of $\mathcal{H}(X)$ coincides with the identity path-component of $\mathcal{H}(X)$ and it is an open normal subgroup of $\mathcal{H}(X)$. If X has empty boundary then the (restricted) evaluation flow $\mathcal{H}_0(X) \curvearrowright X$ is transitive, hence minimal.

The topology of $\mathcal{H}_0(X)$ for compact connected surfaces X was studied by Hamstrom [28–30] (see also [2, Theorem 1]), who proved that $\mathcal{H}_0(X)$ is homeomorphic to

- $\mathbb{T}^1 \times \ell_2$ if X is the disc, the annulus, the Möbius band or the Klein bottle,
- $\mathbb{T}^2 \times \ell_2$ if X is the torus,
- $\mathrm{SO}(3) \times \ell_2$ if X is the sphere or the projective plane,
- ℓ_2 for every other X.

It follows that for surfaces from the fourth item $\mathcal{H}_0(X)$ is contractible.

2.12 Topology of surfaces

Let X be a compact connected manifold of dimension n. The *interior* and the *boundary* of X will be denoted by $\mathrm{Int}(X)$ and $\mathrm{Bd}(X)$, respectively. The interior is open and the boundary is closed, hence $\mathrm{Bd}(X)$ is a compact manifold without boundary and of dimension $n - 1$.

Now let $n = 2$. Then $\mathrm{Bd}(X)$ is a disjoint union of finitely many simple closed curves. If $D \subseteq X$ is a closed topological disc then $\mathrm{Int}(D)$ is open in X by the invariance of domain theorem. If X has empty boundary then $\mathcal{H}_0(X)$ acts transitively on ordered pairs of distinct points of X. Consequently, for all $x, y \in X$ there is a closed topological disc $D \subseteq X$ with $x, y \in \mathrm{Int}(D)$.

Recall that when X is a compact connected surface without boundary, then its *Euler characteristic* $\chi(X)$ is nonnegative precisely when X is either the sphere, the torus, the projective plane or the Klein bottle. Consequently, if $\chi(X) < 0$ then $H_1^w(X) \neq 0$ [46, § 75] and $\mathcal{H}_0(X)$ is contractible as recalled in our Sect. 2.11.

2.13 Quasicoboundaries

Let G be an abelian topological group. A cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ is called a *quasicoboundary* if its values $\mathcal{C}(\gamma, x)$ depend only on $\gamma \in \Gamma$. Equivalently, $\mathcal{C}(\gamma, x) = h(\gamma)$ for some $h \in \mathrm{Hom}(\Gamma, G)$, in which case we write $\mathcal{C} = \mathcal{Q}_h$. Quasicoboundaries form a subgroup $\mathbf{Q}_{\mathcal{F}}(G)$ of $\mathbf{Z}_{\mathcal{F}}(G)$, which is closed when Γ is locally compact.

2.14 Compact abelian extensions

Let G be a compact abelian group and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. The group G acts continuously on $X \times G$ via *vertical rotations* R_g ($g \in G$), each of which is an automorphism of the flow $\mathcal{F}_{\mathcal{C}}$. Since this action is transitive on each fibre of $X \times G$ and every minimal set of $\mathcal{F}_{\mathcal{C}}$ projects onto X, the space $X \times G$ is the union of minimal sets of $\mathcal{F}_{\mathcal{C}}$. Fix $z \in X$ and let M be the minimal set of $\mathcal{F}_{\mathcal{C}}$ containing $(z, 1)$. Denote by $F(\mathcal{C})$ the vertical section of M over z. Then $F(\mathcal{C})$ is the stabilizer of M under the action of G,

$$F(\mathcal{C}) = \{g \in G \colon R_g(M) = M\}, \tag{2.12}$$

hence it is a closed subgroup of G. Since G acts transitively on the set of minimal sets of $\mathcal{F}_{\mathcal{C}}$, $F(\mathcal{C})$ does not depend on M. By [12, Theorem 3.7, p. 63], $F(\mathcal{C})$ is an invariant of cohomology and the assignment $\mathcal{C} \mapsto F(\mathcal{C})$ is covariant functorial in the following sense.

Lemma 2.2 *Let G, H be compact abelian groups, $q \in \operatorname{Hom}(G, H)$ and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then $q\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(H)$ and*

$$F(q\mathcal{C}) = qF(\mathcal{C}).$$

2.15 Contravariant functoriality of cohomology

Let $\mathcal{F}, \mathcal{G}$ be minimal Γ-flows on spaces X, Y, respectively, $h \colon \mathcal{G} \to \mathcal{F}$ be a morphism and G be an abelian topological group. Given $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, consider the map

$$\mathcal{D} \colon \Gamma \times Y \to G, \quad \mathcal{D}(\gamma, y) := \mathcal{C}(\gamma, h(y)).$$

Then $\mathcal{D} \in \mathbf{Z}_{\mathcal{G}}(G)$ and the assignment

$$h_G \colon \mathbf{Z}_{\mathcal{F}}(G) \to \mathbf{Z}_{\mathcal{G}}(G), \quad \mathcal{C} \mapsto \mathcal{D}, \tag{2.13}$$

defines a morphism of groups. Since h is surjective, h_G is a monomorphism. Further, if $\xi \colon X \to G$ is a continuous map then

$$h_G(\operatorname{co}(\xi)) = \operatorname{co}(\xi h). \tag{2.14}$$

Consequently, h_G maps $\mathbf{B}_{\mathcal{F}}(G)$ into $\mathbf{B}_{\mathcal{G}}(G)$, thus inducing a morphism $\mathbf{H}_{\mathcal{F}}(G) \to \mathbf{H}_{\mathcal{G}}(G)$ which we shall denote by the same symbol h_G.

2.16 Induced morphisms between cohomology groups

Let G, K be locally compact abelian groups and $q \colon G \to K$ be a topological morphism. Then q gives rise to a morphism

$$\varphi \colon \mathbf{Z}_{\mathcal{F}}(G) \to \mathbf{Z}_{\mathcal{F}}(K), \quad \mathcal{C} \mapsto q\mathcal{C},$$

which is topological when Γ is locally compact. For every continuous map $\xi \colon X \to G$,

$$q \operatorname{co}(\xi) = \operatorname{co}(q\xi). \tag{2.15}$$

Consequently, φ maps $\mathbf{B}_{\mathcal{F}}(G)$ into $\mathbf{B}_{\mathcal{F}}(K)$, thus inducing a morphism $\mathbf{H}_{\mathcal{F}}(G) \to \mathbf{H}_{\mathcal{F}}(K)$.

The following particular case will be of special interest to us.

Definition 2.3 We denote by $\operatorname{Exp}_{\mathcal{F}}$ (or Exp for short) the morphism $\mathbf{Z}_{\mathcal{F}}(\mathbb{R}) \to \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ induced by the exponential $\exp \colon \mathbb{R} \to \mathbb{T}$ and call it the *induced exponential* of $\mathcal{F}$.

It should be noted that $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ is a real linear space and $\mathbf{B}_{\mathcal{F}}(\mathbb{R})$ is its linear subspace, so $\mathbf{H}_{\mathcal{F}}(\mathbb{R})$ is also linear. If Γ is locally compact then the compact-open topology turns $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ into a locally convex linear topological space. Therefore $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ is both contractible and locally contractible.

2.17 Basic properties of the induced exponential

The purpose of this section is to collect some elementary properties of the induced exponential Exp. Other properties of Exp will be discussed in due course.

Lemma 2.4 *In the notation of Sect. 2.13 we have*

$$\ker(\mathrm{Exp}) = \mathbf{Q}_{\mathcal{F}}(\mathbb{Z}). \tag{2.16}$$

Moreover, the following statements hold.

(i) *If Γ is locally compact and compactly generated then* $\ker(\mathrm{Exp})$ *is discrete in* $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$.
(ii) *If Γ is connected then* $\ker(\mathrm{Exp}) = \{0\}$, *hence* Exp *is a monomorphism.*

Proof Inclusion "$\supseteq$" from (2.16) is clear. To verify the converse inclusion, let $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ satisfy $\exp \mathcal{C} = 1$. Then $\mathcal{C}$ takes its values in $\mathbb{Z}$. Since X is connected and $\mathbb{Z}$ is discrete, it follows that $\mathcal{C}(\gamma, x)$ depends only on $\gamma \in \Gamma$ and not on $x \in X$. Thus $\mathcal{C} = \mathcal{Q}_h$ for some $h \in \mathrm{Hom}(\Gamma, \mathbb{Z})$.

Now assume that Γ is locally compact and generated by a compact set K. Recall our notation (2.1) and set

$$\mathcal{V} := [K \times X, (-1, 1)] \subseteq \mathbf{Z}_{\mathcal{F}}(\mathbb{R}).$$

Clearly, $\mathcal{V}$ is an identity neighbourhood in $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ and we verify (i) by showing that $\ker(\mathrm{Exp}) \cap \mathcal{V} = \{0\}$. So use (2.16) and choose $\mathcal{C} \in \mathbf{Q}_{\mathcal{F}}(\mathbb{Z}) \cap \mathcal{V}$. Write $\mathcal{C} = \mathcal{Q}_h$ with $h \in \mathrm{Hom}(\Gamma, \mathbb{Z})$. Then $h(K) = \{0\}$ and since K generates Γ, $h = 0$. Thus $\mathcal{C} = 0$ indeed.

Finally, if Γ is connected then $\mathbf{Q}_{\mathcal{F}}(\mathbb{Z}) = \{0\}$, hence (ii) follows from (2.16). □

Lemma 2.5 *The group* $\mathrm{im}(\mathrm{Exp})$ *is a direct summand in* $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ *with a complementary summand isomorphic to* $\mathrm{coker}(\mathrm{Exp}) = \mathbf{Z}_{\mathcal{F}}(\mathbb{T}) / \mathrm{im}(\mathrm{Exp})$.

Proof Being a real linear space, $\mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ is a divisible group, hence so is $\mathrm{im}(\mathrm{Exp})$. Since divisible subgroups of abelian groups are direct summands, the result follows. □

Lemma 2.5 now allows the following definition.

Definition 2.6 For every minimal flow $\mathcal{F}$ we fix a complementary summand of $\mathrm{im}(\mathrm{Exp})$ in $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ and denote it by $\mathbf{A}_{\mathcal{F}}$, thus having

$$\mathbf{Z}_{\mathcal{F}}(\mathbb{T}) = \mathrm{im}(\mathrm{Exp}) \oplus \mathbf{A}_{\mathcal{F}}. \tag{2.17}$$

Lemma 2.7 *If $h \colon \mathcal{G} \to \mathcal{F}$ is a morphism of minimal flows then*

$$\mathrm{Exp}_{\mathcal{G}} h_{\mathbb{R}} = h_{\mathbb{T}} \mathrm{Exp}_{\mathcal{F}}$$

(see (2.13) and Fig. 1).

Proof For every $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$,

$$\mathrm{Exp}_{\mathcal{G}} h_{\mathbb{R}}(\mathcal{C}) = \mathrm{Exp}_{\mathcal{G}}\big(\mathcal{C}(\mathrm{id}_\Gamma \times h)\big) = \exp \mathcal{C}(\mathrm{id}_\Gamma \times h) = \mathrm{Exp}_{\mathcal{F}}(\mathcal{C})(\mathrm{id}_\Gamma \times h) = h_{\mathbb{T}} \mathrm{Exp}_{\mathcal{F}}(\mathcal{C}).$$

□

Fig. 1 The induced exponential and morphisms of cocycle groups

$$\begin{tikzcd}
\mathbf{Z}_{\mathcal{F}}(\mathbb{R}) \arrow[r, "h_{\mathbb{R}}"] \arrow[d, "\mathrm{Exp}_{\mathcal{F}}"'] & \mathbf{Z}_{\mathcal{G}}(\mathbb{R}) \arrow[d, "\mathrm{Exp}_{\mathcal{G}}"] \\
\mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \arrow[r, "h_{\mathbb{T}}"] & \mathbf{Z}_{\mathcal{G}}(\mathbb{T})
\end{tikzcd}$$

3 One-dimensional compactifications of real cocycles

This section is devoted to compactifications of real cocycles by means of characters $\chi \in \mathbb{R}^*$. To every minimal flow $\mathcal{F}$ and every cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ we associate subgroups $\pi^1(\mathcal{F})$ of $\pi^1(X)$ and $\mathcal{C}^\times$ of $\mathbb{R}^*$ (see Definitions 3.10 and 3.6 below), and present their basic properties. Among results of this section the following three deserve special attention because of their direct use in the proofs of our three main theorems.

- Proposition 3.16 which binds into an exact sequence the cohomology groups $\mathbf{H}_{\mathcal{F}}(\mathbb{R})$, $\mathbf{H}_{\mathcal{F}}(\mathbb{T})$, the first cohomotopy group $\pi^1(X)$ and its subgroup $\pi^1(\mathcal{F})$ and the group $\mathbf{A}_{\mathcal{F}}$ from Definition 2.6.
- Theorem 3.17 which connects $\mathcal{C}^\times$ with $\pi^1(\mathcal{F})$ by means of the morphism $\nu_{\mathcal{C}}$ defined by (3.6).
- Lemma 3.21 which shows that morphisms of real cohomology groups induced by morphisms of minimal flows are injective.

3.1 Nonminimal compactifications and coboundaries

Given a locally compact abelian group G and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, consider the map

$$\mathcal{C}^* \colon G^* \to \mathbf{Z}_{\mathcal{F}}(\mathbb{T}), \quad \chi \mapsto \chi\mathcal{C}. \tag{3.1}$$

Clearly, $\mathcal{C}^*$ is a morphism of groups.

Definition 3.1 For every locally compact abelian group G and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, we define the group

$$\mathcal{C}^\perp := (\mathcal{C}^*)^{-1}(\mathbf{B}_{\mathcal{F}}(\mathbb{T})) = \{\chi \in G^* \colon \chi\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})\}. \tag{3.2}$$

Definition 3.1 is partly motivated by the following Lemmas 3.2 and 3.4 which show that if G is compact then $\mathcal{C}^\perp$ detects minimal elements of $\mathbf{Z}_{\mathcal{F}}(G)$ as well as coboundaries. We will also use it in our proof of Theorem 3.17. (For the definition of the group $F(\mathcal{C})$ used below, see Sect. 2.14.)

Lemma 3.2 *Let G be a compact abelian group and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then the following conditions are equivalent:*

(a) *$\mathcal{C}$ is minimal,*
(b) *$F(\mathcal{C}) = G$,*
(c) *$\mathcal{C}^\perp = \{1\}$.*

Proof Conditions (a) and (b) are equivalent by [12, Theorem 3.7, p. 63]. The equivalence of (a) and (c) follows from [12, Corollary 3.8, p. 63]. □

Remark 3.3 Since $F(\mathcal{C})$ is a cohomology invariant by [12, Theorem 3.7, p. 63], it follows from Lemma 3.2 that so is minimality of $\mathcal{C}$.

Lemma 3.4 *Let G be a compact abelian group and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then the following conditions are equivalent:*

(α) $\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(G)$,
(β) $F(\mathcal{C}) = \{1\}$,
(γ) $\mathcal{C}^{\perp} = G^*$.

Proof The equivalence of (α) and (β) follows from [12, Theorem 3.7, p. 63]. Conditions (α) and (γ) are quivalent by [12, Corollary 3.8, p. 63]. ◻

Remark 3.5 Let us mention that an analogy of the equivalence of conditions (α) and (γ) is also of interest in the measure-theoretic setting [37, 44]. We shall see in Lemma 3.19 that the equivalence in the topological setting extends to cocycles with values in $\mathbb{R}$ under the assumption that X is metrizable. (For flows on nonmetrizable spaces this extension is not valid, see Remark 3.20.)

In our proof of Theorem 1.3 we shall compactify a real cocycle by means of characters and ask one of these compactifications to be minimal. For this reason we will try to understand the structure of all nonminimal compactifications of a given cocycle $\mathcal{C}$.

Definition 3.6 Given a locally compact abelian group G and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, set

$$\mathcal{C}^{\times} := \{\chi \in G^* \colon \mathcal{C}^*(\chi) \text{ is not minimal}\}. \tag{3.3}$$

In order to show how $\mathcal{C}^{\times}$ is related to $\mathcal{C}^{\perp}$, we fix a notation. For a subgroup B of an abelian group A we let

$$D_A(B) := \{a \in A \colon da \in B \text{ for some integer } d \geq 1\}. \tag{3.4}$$

Clearly, $D_A(B)$ is a subgroup of A containing B and we have equality of ranks $\mathrm{r}(D_A(B)) = \mathrm{r}(B)$.

Lemma 3.7 *Let G be a locally compact abelian group and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then*

$$\mathcal{C}^{\times} = D_{G^*}(\mathcal{C}^{\perp}). \tag{3.5}$$

Consequently, $\mathcal{C}^{\times}$ is a subgroup of G^* containing $\mathcal{C}^{\perp}$ and $\mathrm{r}(\mathcal{C}^{\times}) = \mathrm{r}(\mathcal{C}^{\perp})$.

Proof Fix $\chi \in G^*$. By Lemma 3.2 we have $\chi \in \mathcal{C}^{\times}$ if, and only if, $F(\chi\mathcal{C})$ is a finite subgroup of $\mathbb{T}$. This occurs precisely when $\kappa_d F(\chi\mathcal{C}) = \{1\}$ for some $d \geq 1$, where κ_d denotes the d-endomorphism of $\mathbb{T}$. According to Lemma 2.2, this is equivalent to $F(\kappa_d \chi\mathcal{C}) = \{1\}$, hence also to $\chi^d\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})$ by Lemma 3.4. This amounts to $\chi^d \in \mathcal{C}^{\perp}$ for some $d \geq 1$, which is equivalent with $\chi \in D_{G^*}(\mathcal{C}^{\perp})$. ◻

Now we show that the assignments $\mathcal{C} \mapsto \mathcal{C}^{\perp}$ and $\mathcal{C} \mapsto \mathcal{C}^{\times}$ are contravariant functorial. (Recall that q^* denotes the Pontryagin dual of a topological morphism q.)

Lemma 3.8 *Let G, H be locally compact abelian groups, $q \in \mathrm{Hom}(G, H)$ and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then*

$$(q\mathcal{C})^* = \mathcal{C}^* q^*, \quad (q\mathcal{C})^{\perp} = (q^*)^{-1}(\mathcal{C}^{\perp}) \ \textit{ and } \ (q\mathcal{C})^{\times} = (q^*)^{-1}(\mathcal{C}^{\times})$$

(see Fig. 2).

Fig. 2 Morphisms induced by cocycles

$$\begin{tikzcd}
 & G \arrow[dd, "q"] & & G^* \arrow[dr, "\mathcal{C}^*"] & \\
\mathcal{F} \arrow[ur, "\mathcal{C}"] \arrow[dr, "q\mathcal{C}"'] & & & & \mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \\
 & H & & H^* \arrow[uu, "q^*"] \arrow[ur, "(q\mathcal{C})^*"'] &
\end{tikzcd}$$

Proof For every $\chi \in H^*$,

$$(q\mathcal{C})^*(\chi) = \chi(q\mathcal{C}) = (\chi q)\mathcal{C} = \mathcal{C}^*(\chi q) = \mathcal{C}^* q^*(\chi).$$

Consequently,

$$(q\mathcal{C})^{\perp} = ((q\mathcal{C})^*)^{-1}(\mathbf{B}_{\mathcal{F}}(\mathbb{T})) = (q^*)^{-1}((\mathcal{C}^*)^{-1}(\mathbf{B}_{\mathcal{F}}(\mathbb{T}))) = (q^*)^{-1}(\mathcal{C}^{\perp}).$$

Finally, if $\chi \in H^*$ then $\chi \in (q\mathcal{C})^{\times}$ if, and only if, $(\chi q)\mathcal{C}$ is not minimal. This is equivalent to $q^*(\chi) \in \mathcal{C}^{\times}$, hence also to $\chi \in (q^*)^{-1}(\mathcal{C}^{\times})$. □

3.2 The group $\pi^1(\mathcal{F})$

Given $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$, recall our notation (3.2) and consider the map

$$\nu_{\mathcal{C}} \colon \mathcal{C}^{\perp} \to \pi^1(X), \quad \nu_{\mathcal{C}}(\chi) := \xi, \text{ where } \chi\mathcal{C} = \operatorname{co}(\xi). \tag{3.6}$$

By our earlier convention we use preferably base point preserving maps as transfer functions of coboundaries, so $\nu_{\mathcal{C}}$ is well defined. However, since all possible transfer functions of a given coboundary differ by additive constants, hence are mutually homotopic, any transfer function of $\chi\mathcal{C}$ can be used in the definition of $\nu_{\mathcal{C}}(\chi)$.

The map $\nu_{\mathcal{C}}$ defined above (which in fact is a morphism of groups as shown below) will help us understand how the topology of X (or of $\mathcal{F}$) reflects in the structure of the group $\mathcal{C}^{\times}$ from (3.3) (see Theorem 3.17).

Proposition 3.9 *If $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ then $\nu_{\mathcal{C}}$ is a morphism of groups depending only on the cohomology class of $\mathcal{C}$. Moreover, if Γ is connected then the cokernel of $\nu_{\mathcal{C}}$,*

$$\operatorname{coker}(\nu_{\mathcal{C}}) := \pi^1(X)/\operatorname{im}(\nu_{\mathcal{C}}),$$

is torsion-free.

Proof We proceed in three steps.

STEP 1. Given $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$, we show that $\nu_{\mathcal{C}}$ is a morphism of groups.

Fix $\chi, \Upsilon \in \mathcal{C}^{\perp}$ and write $\chi\mathcal{C} = \operatorname{co}(\xi)$, $\Upsilon\mathcal{C} = \operatorname{co}(\zeta)$. Then $(\chi \cdot \Upsilon)\mathcal{C} = (\chi\mathcal{C}) \cdot (\Upsilon\mathcal{C}) = \operatorname{co}(\xi \cdot \zeta)$, hence

$$\nu_{\mathcal{C}}(\chi \cdot \Upsilon) = \xi \cdot \zeta = \nu_{\mathcal{C}}(\chi) \cdot \nu_{\mathcal{C}}(\Upsilon).$$

STEP 2. Given $\mathcal{C}, \mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$, we show that $\nu_{\mathcal{C}} = \nu_{\mathcal{D}}$ whenever $\mathcal{C}$ and $\mathcal{D}$ are cohomologous.

Let $\mathcal{D} = \mathcal{C} + \operatorname{co}(\eta)$. Then for every character $\chi \in \mathbb{R}^*$, $\chi\mathcal{D} = (\chi\mathcal{C}) \cdot \operatorname{co}(\chi\eta)$, so $\chi\mathcal{D} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})$ if, and only if, $\chi\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})$. Thus it follows that $\mathcal{C}^{\perp} = \mathcal{D}^{\perp}$. Further, if $\chi \in \mathcal{C}^{\perp}$ and $\chi\mathcal{C} = \operatorname{co}(\xi)$ then $\chi\mathcal{D} = \operatorname{co}(\xi \cdot (\chi\eta))$. Since $\chi\eta$ represents the neutral element of $\pi^1(X)$, we conclude with $\nu_{\mathcal{C}}(\chi) = \xi = \nu_{\mathcal{D}}(\chi)$.

STEP 3. Let Γ be connected. We show that $\operatorname{coker}(\nu_{\mathcal{C}})$ is torsion-free.

Fix a continuous map $\xi\colon X \to \mathbb{T}$ and assume that $\xi^d \in \mathrm{im}(\nu_{\mathcal{C}})$ for some integer $d \geq 1$; we show that $\xi \in \mathrm{im}(\nu_{\mathcal{C}})$. Choose $\chi \in \mathcal{C}^{\perp}$ with $\xi^d = \nu_{\mathcal{C}}(\chi)$. Then $\chi\mathcal{C} = \mathrm{co}(\zeta)$ for an appropriate continuous map $\zeta\colon X \to \mathbb{T}$ homotopic to ξ^d. By replacing ξ with an appropriate map homotopic to it, we may assume that $\zeta = \xi^d$, hence $\chi\mathcal{C} = \mathrm{co}(\xi^d)$. Since $\mathbb{R}^*$ is divisible, $\chi = \lambda^d$ for some $\lambda \in \mathbb{R}^*$. Consequently, $\left[(\lambda\mathcal{C}) \cdot \mathrm{co}(\xi)^{-1}\right]^d = 1$, hence $(\lambda\mathcal{C}) \cdot \mathrm{co}(\xi)^{-1} = 1$ by connectedness of Γ and X. Thus $\lambda\mathcal{C} = \mathrm{co}(\xi)$ and we conclude with $\xi = \nu_{\mathcal{C}}(\lambda) \in \mathrm{im}(\nu_{\mathcal{C}})$. □

The following definition is central to our work. The group $\pi^1(\mathcal{F})$ defined here will be an important tool for our study of algebraic topology of universal minimal flows in Sect. 5.

Definition 3.10 Given a minimal flow $\mathcal{F}$, set

$$\pi^1(\mathcal{F}) := \{\xi \in \pi^1(X)\colon\ \mathrm{co}(\xi) \in \mathrm{im}(\mathrm{Exp})\}. \tag{3.7}$$

The next proposition justifies that Definition 3.10 is correct and relates $\pi^1(\mathcal{F})$ to morphisms $\nu_{\mathcal{C}}$ from (3.6).

Proposition 3.11 *For every continuous map* $\xi\colon X \to \mathbb{T}$, *the incidence* $\mathrm{co}(\xi) \in \mathrm{im}(\mathrm{Exp})$ *depends only on the homotopy class of* ξ. *Further,* $\pi^1(\mathcal{F})$ *is a subgroup of* $\pi^1(X)$ *and*

$$\pi^1(\mathcal{F}) = \bigcup_{\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})} \mathrm{im}(\nu_{\mathcal{C}}). \tag{3.8}$$

Thus $\pi^1(\mathcal{F})$ *is the smallest among the subgroups of* $\pi^1(X)$ *containing* $\mathrm{im}(\nu_{\mathcal{C}})$ *for every* $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$.

Proof Let $\xi, \zeta\colon X \to \mathbb{T}$ and $\eta\colon X \to \mathbb{R}$ be continuous maps with $\xi = \zeta \cdot \exp\eta$. Then $\mathrm{co}(\xi) = \mathrm{co}(\zeta) \cdot \mathrm{Exp}(\mathrm{co}(\eta))$, so $\mathrm{co}(\xi)$ belongs to $\mathrm{im}(\mathrm{Exp})$ if, and only if, so does $\mathrm{co}(\zeta)$. The fact that $\pi^1(\mathcal{F})$ is a subgroup of $\pi^1(X)$ follows from the fact that $\mathrm{im}(\mathrm{Exp})$ is a subgroup of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$. We finish the proof by verifying (3.8). To verify inclusion "$\supseteq$", fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ together with $\chi_t \in \mathcal{C}^{\perp}$ (see (2.6)) and write $\chi_t\mathcal{C} = \mathrm{co}(\xi)$. Then $\mathrm{co}(\xi) = \mathrm{Exp}(t\mathcal{C}) \in \mathrm{im}(\mathrm{Exp})$, hence $\nu_{\mathcal{C}}(\chi_t) = \xi \in \pi^1(\mathcal{F})$. To verify inclusion "$\subseteq$", fix $\xi \in \pi^1(\mathcal{F})$ and choose $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $\mathrm{co}(\xi) = \mathrm{Exp}(\mathcal{C})$. Then $\chi_1 \in \mathcal{C}^{\perp}$ and $\xi = \nu_{\mathcal{C}}(\chi_1) \in \mathrm{im}(\nu_{\mathcal{C}})$. □

Now we show that $\pi^1(\mathcal{F})$ is contravariant functorial in $\mathcal{F}$, a useful fact that we shall use repeatedly without explicit reference.

Proposition 3.12 *Let* Γ *be a topological group. Then the assignment* $\mathcal{F} \mapsto \pi^1(\mathcal{F})$ *from the category of minimal* Γ*-flows (with compact and connected phase spaces) into the category of torsion-free abelian groups is contravariant functorial.*

Proof Let $\mathcal{F}, \mathcal{G}$ be minimal Γ-flows and $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism. Denote by X and Y the phase spaces of $\mathcal{F}$ and $\mathcal{G}$, respectively. It suffices to show that the induced morphism $h^{\flat}\colon \pi^1(X) \to \pi^1(Y)$ restricts to a morphism $\pi^1(\mathcal{F}) \to \pi^1(\mathcal{G})$. To this end, fix $\xi \in \pi^1(\mathcal{F})$ and choose $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $\mathrm{co}(\xi) = \mathrm{Exp}_{\mathcal{F}}(\mathcal{C})$. By (2.14) and Lemma 2.7,

$$\mathrm{co}(\xi h) = h_{\mathbb{T}}(\mathrm{co}(\xi)) = h_{\mathbb{T}}\mathrm{Exp}_{\mathcal{F}}(\mathcal{C}) = \mathrm{Exp}_{\mathcal{G}} h_{\mathbb{R}}(\mathcal{C}) \in \mathrm{im}(\mathrm{Exp}_{\mathcal{G}}),$$

so $h^{\flat}(\xi) = \xi h \in \pi^1(\mathcal{G})$. □

The next corollary immediately follows.

Corollary 3.13 *If $h\colon \mathcal{G} \to \mathcal{F}$ is an isomorphism of minimal flows then $h^\flat\colon \pi^1(\mathcal{F}) \to \pi^1(\mathcal{G})$ is an isomorphism of groups.*

Now we relate the group $\pi^1(\mathcal{F})$ to the group $\mathbf{A}_\mathcal{F}$ from Definition 2.6.

Proposition 3.14 *The quotient group $\pi^1(X)/\pi^1(\mathcal{F})$ is isomorphic to a subgroup of the group $\mathbf{A}_\mathcal{F}$ defined by (2.17). If Γ is connected then $\pi^1(X)/\pi^1(\mathcal{F})$ is torsion-free.*

Proof Consider the map

$$\varphi\colon \mathbf{B}_\mathcal{F}(\mathbb{T}) \to \pi^1(X), \quad \mathrm{co}(\xi) \mapsto \xi.$$

Since transfer functions of a given coboundary are mutually homotopic, φ is well defined. Further, φ is an epimorphism of groups with

$$\varphi^{-1}(\pi^1(\mathcal{F})) = \mathbf{B}_\mathcal{F}(\mathbb{T}) \cap \mathrm{im}(\mathrm{Exp}),$$

so $\pi^1(X)/\pi^1(\mathcal{F})$ is isomorphic to $\mathbf{B}_\mathcal{F}(\mathbb{T})/(\mathbf{B}_\mathcal{F}(\mathbb{T}) \cap \mathrm{im}(\mathrm{Exp}))$. Since the latter group may be identified with a subgroup of $\mathbf{Z}_\mathcal{F}(\mathbb{T})/\mathrm{im}(\mathrm{Exp}) \cong \mathbf{A}_\mathcal{F}$ (see Lemma 2.5), the first claim of the proposition follows. As for the second claim, if Γ is connected then $\mathbf{Z}_\mathcal{F}(\mathbb{T})$ is torsion-free [12, Theorem 4.10, p. 95], hence so is $\mathbf{A}_\mathcal{F} \subseteq \mathbf{Z}_\mathcal{F}(\mathbb{T})$ and it suffices to use the first claim of the proposition. □

Example 3.15 Let Γ be a connected Lie group and X be a manifold. We show that

$$\mathrm{r}(\pi^1(\mathcal{F})) = \mathrm{r}(H_1^w(X)) - \mathrm{r}(H_1^w(\mathcal{F}))$$

with $H_1^w(\mathcal{F})$ defined by (2.9). Write $n = \mathrm{r}(H_1^w(\mathcal{F}))$ and $n + m = \mathrm{r}(H_1^w(X))$. Given $\xi \in \pi^1(X)$, we have $\mathrm{co}(\xi) \in \mathrm{im}(\mathrm{Exp})$ if, and only if, $\mathrm{co}(\xi)$ lifts across exp to a continuous map $\mathcal{C}\colon \Gamma \times X \to \mathbb{R}$ with $\mathcal{C}(1, z) = 0$ [12, Lemma 3.36, p. 84]. Since $\mathbb{R}$ is simply connected, this is equivalent to $\mathrm{co}(\xi)^\sharp = 0$ [46, Lemma 79.1, p. 478]. Moreover, by [12, Lemma 2.17, p. 55], this occurs precisely when $\xi^\sharp(H_1^w(\mathcal{F})) = 0$. Consequently, under the isomorphism

$$\sigma\colon \pi^1(X) \to \mathrm{Hom}(H_1^w(X), H_1^w(\mathbb{T})), \quad \xi \mapsto \xi^\sharp,$$

from Sect. 2.10, $\pi^1(\mathcal{F})$ corresponds to the subgroup formed by the morphisms vanishing on $H_1^w(\mathcal{F})$. This means that $\pi^1(\mathcal{F})$ is a free abelian group with rank m.

The following proposition is inspired by [12, Theorem 3.43, p. 88]. It will serve as a key instrument in our proof of Theorem 1.2.

Proposition 3.16 *If Γ is connected then there is an exact sequence*

$$0 \to \pi^1(X) \xrightarrow{\psi} \mathbf{H}_\mathcal{F}(\mathbb{R}) \oplus \mathbf{A}_\mathcal{F} \to \mathbf{H}_\mathcal{F}(\mathbb{T}) \to 0,$$

such that

$$\psi^{-1}(\mathbf{H}_\mathcal{F}(\mathbb{R})) = \pi^1(\mathcal{F}). \tag{3.9}$$

Proof Set $\mathbf{E} := \mathrm{Exp}(\mathbf{B}_\mathcal{F}(\mathbb{R}))$. Since Exp is a monomorphism by Lemma 2.4, $\mathbf{H}_\mathcal{F}(\mathbb{R}) = \mathbf{Z}_\mathcal{F}(\mathbb{R})/\mathbf{B}_\mathcal{F}(\mathbb{R})$ may be identified with $\mathbf{H} := \mathrm{im}(\mathrm{Exp})/\mathbf{E}$. Consider the morphism of groups

$$\varphi\colon C(X, \mathbb{T}) \to \mathbf{Z}_\mathcal{F}(\mathbb{T}), \quad \xi \mapsto \mathrm{co}(\xi).$$

We have $\mathrm{co}(\exp \vartheta) = \exp \mathrm{co}(\vartheta)$ for every continuous map $\vartheta\colon X \to \mathbb{R}$, hence the identity component $C(X, \mathbb{T})_0$ of $C(X, \mathbb{T})$ is mapped by φ onto $\mathbf{E}$. Moreover, since transfer functions of coboundaries are unique up to additive constants, we have $\varphi^{-1}(E) = C(X, \mathbb{T})_0$. Thus φ gives rise to a monomorphism

$$\psi : \pi^1(X) \to \mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\mathbf{E},$$

which takes the form

$$\psi : \pi^1(X) \to [\mathrm{im}(\mathrm{Exp}) \oplus \mathbf{A}_{\mathcal{F}}]/\mathbf{E} \cong \mathbf{H} \oplus \mathbf{A}_{\mathcal{F}}$$

according to Lemma 2.5. Further, given $\xi \in C(X, \mathbb{T})$, we have $\psi(\xi) \in \mathbf{H}$ if, and only if, $\mathrm{co}(\xi) \in \mathrm{im}(\mathrm{Exp})$, which is equivalent to $\xi \in \pi^1(\mathcal{F})$ by Definition 3.10. Thus follows (3.9). Finally, $\mathrm{im}(\psi)$ coincides with the subgroup $\mathbf{B}_{\mathcal{F}}(\mathbb{T})/\mathbf{E}$ of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\mathbf{E}$ and the corresponding quotient group is isomorphic to $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\mathbf{B}_{\mathcal{F}}(\mathbb{T}) = \mathbf{H}_{\mathcal{F}}(\mathbb{T})$. □

3.3 The structure of nonminimal compactifications

The following theorem is our main result in this section. It shows how the topology of X reflects in the structure of $\mathcal{C}^{\perp}$ and $\mathcal{C}^{\times}$, and binds these two groups with $\pi^1(\mathcal{F})$ by means of $\nu_{\mathcal{C}}$. (For definitions of these four notions see (3.2), (3.3), (3.7) and (3.6), respectively.)

Theorem 3.17 *Given $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$, the following statements hold.*

(1) *If $\mathcal{C}$ is a coboundary then $\mathcal{C}^{\times} = \mathcal{C}^{\perp} = \mathbb{R}^*$ and $\nu_{\mathcal{C}} = 0$.*
(2) *If $\mathcal{C}$ is cohomologous to a quasicoboundary $\mathcal{Q}_h$ with $0 \neq h \in \mathrm{Hom}(\Gamma, \mathbb{R})$ having discrete image then $\ker(\nu_{\mathcal{C}})$ is infinite cyclic and*

$$\mathrm{r}(\mathcal{C}^{\times}) \leq \mathrm{r}(\pi^1(\mathcal{F})) + 1.$$

(3) *In all the other cases $\nu_{\mathcal{C}}$ is a monomorphism and*

$$\mathrm{r}(\mathcal{C}^{\times}) \leq \mathrm{r}(\pi^1(\mathcal{F})).$$

Proof We divide the proof into four steps.

Step 1. We verify (1).

First, since $\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{R})$, we have $\chi\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})$ for every $\chi \in \mathbb{R}^*$, hence $\mathcal{C}^{\perp} = \mathbb{R}^*$. Consequently, $\mathcal{C}^{\times} = \mathbb{R}^*$. Further, if $\mathcal{C} = \mathrm{co}(\eta)$ and $\chi \in \mathbb{R}^*$ then $\chi\mathcal{C} = \mathrm{co}(\chi\eta)$ and since $\chi\eta$ represents the neutral element of $\pi^1(X)$, $\nu_{\mathcal{C}}(\chi) = 0$.

Step 2. We verify the first assertion from (2).

Write $\mathcal{C} = \mathcal{Q}_h + \mathrm{co}(\eta)$ with η base point preserving and choose $1 \neq \Upsilon \in \mathbb{R}^*$ with $\ker(\Upsilon) = \mathrm{im}(h)$. It suffices to show that

$$\ker(\nu_{\mathcal{C}}) = \langle \Upsilon \rangle,$$

where $\langle \Upsilon \rangle$ denotes the subgroup of $\mathbb{R}^*$ generated by Υ.

To verify "$\supseteq$", fix $k \in \mathbb{Z}$ and write $\chi = \Upsilon^k$. Then

$$\ker(\chi) = \ker(\Upsilon^k) \supseteq \ker(\Upsilon) = \mathrm{im}(h),$$

hence $\chi h = 1$. Consequently,

$$\chi\mathcal{C} = \chi\mathcal{Q}_h \cdot \chi\,\mathrm{co}(\eta) = \mathcal{Q}_{\chi h} \cdot \mathrm{co}(\chi\eta) = \mathrm{co}(\chi\eta),$$

meaning that $\chi \in \mathcal{C}^{\perp}$ and $\nu_{\mathcal{C}}(\chi) = 0$.

To verify "$\subseteq$", fix $1 \neq \chi \in \ker(\nu_{\mathcal{C}})$ and write $\chi\mathcal{C} = \mathrm{co}(\xi)$ with ξ base point preserving. Since $\nu_{\mathcal{C}}(\chi) = 0$ and $\chi \neq 1$, we have $\xi = \chi\vartheta$ for some continuous base point preserving map $\vartheta \colon X \to \mathbb{R}$. Then

$$\chi(\mathcal{Q}_h + \mathrm{co}(\eta)) = \chi\mathcal{C} = \mathrm{co}(\xi) = \chi\,\mathrm{co}(\vartheta),$$

hence

$$\chi(\mathcal{Q}_h + \mathrm{co}(\eta - \vartheta)) = 1. \tag{3.10}$$

Since X is connected and $\ker(\chi)$ is discrete, we infer that $\mathcal{Q}_h + \mathrm{co}(\eta - \vartheta)$ depends only on $\gamma \in \Gamma$ and not on $x \in X$. Consequently,

$$\mathcal{Q}_h + \mathrm{co}(\eta - \vartheta) = \mathcal{Q}_k$$

for some $k \in \mathrm{Hom}(\Gamma, \mathbb{R})$. Then $\mathcal{Q}_{k-h} = \mathrm{co}(\eta - \vartheta)$ and since coboundaries are bounded, $k = h$ and $\eta = \vartheta$. Consequently, $\chi h = 1$ on the account of (3.10). Thus

$$\ker(\chi) \supseteq \mathrm{im}(h) = \ker(\Upsilon),$$

meaning that $\chi \in \langle\Upsilon\rangle$.

STEP 3. We verify the second assertion from (2). By Lemma 3.7, we may consider $\mathcal{C}^{\perp}$ instead of $\mathcal{C}^{\times}$.

We have a short exact sequence

$$0 \to \ker(\nu_{\mathcal{C}}) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}^{\perp} \xrightarrow{\nu_{\mathcal{C}}} \mathrm{im}(\nu_{\mathcal{C}}) \to 0,$$

where $\mu_{\mathcal{C}}$ is the inclusion morphism, which leads to

$$\mathrm{r}(\mathcal{C}^{\perp}) = \mathrm{r}(\mathrm{im}(\nu_{\mathcal{C}})) + \mathrm{r}(\ker(\nu_{\mathcal{C}})).$$

By virtue of Step 2, $\mathrm{r}(\ker(\nu_{\mathcal{C}})) = 1$ and by Proposition 3.11, $\mathrm{r}(\mathrm{im}(\nu_{\mathcal{C}})) \leq \mathrm{r}(\pi^1(\mathcal{F}))$, so $\mathrm{r}(\mathcal{C}^{\perp}) \leq \mathrm{r}(\pi^1(\mathcal{F})) + 1$ indeed.

STEP 4. We verify (3).

We begin by proving that $\nu_{\mathcal{C}}$ is monic, proceeding by contradiction. So assume that $1 \neq \chi \in \ker(\nu_{\mathcal{C}})$ and write $\chi\mathcal{C} = \mathrm{co}(\xi)$. Since $\nu_{\mathcal{C}}(\chi) = 0$, we have $\xi = \chi\vartheta$ for some continuous map $\vartheta \colon X \to \mathbb{R}$. Then $\chi(\mathcal{C} - \mathrm{co}(\vartheta)) = 1$, hence by connectedness of X, $\mathcal{C} = \mathrm{co}(\vartheta) + \mathcal{Q}_h$ for some $h \in \mathrm{Hom}(\Gamma, \mathbb{R})$ with $\mathrm{im}(h) \subseteq \ker(\chi)$. By our assumptions on $\mathcal{C}$, we infer that $h = 0$, hence $\mathcal{C} = \mathrm{co}(\vartheta)$, a contradiction.

To verify the second assertion from (3), we employ Lemma 3.7, Proposition 3.11 and the fact that $\nu_{\mathcal{C}}$ is a monomorphism to obtain

$$\mathrm{r}(\mathcal{C}^{\times}) = \mathrm{r}(\mathcal{C}^{\perp}) = \mathrm{r}(\mathrm{im}(\nu_{\mathcal{C}})) \leq \mathrm{r}(\pi^1(\mathcal{F})).$$

□

The following corollary of Theorem 3.17 will be used in our proof of Theorem 1.3, as indicated in the first paragraph of Sect. 5.4.

Corollary 3.18 *If X is metrizable and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ is not a coboundary then $\mathcal{C}^{\times}$ is countable.*

Proof Since $\pi^1(X)$ is countable by the metrizability assumption, parts (2) and (3) of Theorem 3.17 yield

$$\mathrm{r}(\mathcal{C}^{\times}) \leq \mathrm{r}(\pi^1(\mathcal{F})) + 1 \leq \mathrm{card}(\pi^1(\mathcal{F})) + 1 \leq \aleph_0 + 1 = \aleph_0.$$

The countability of $\mathcal{C}^{\times} \subseteq \mathbb{R}^*$ now follows. □

It is well known that a cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ is a coboundary if, and only if, it is bounded (see (a)–(c) below). It turns out that when X is metrizable then $\mathcal{C}$ is a coboundary if, and only if, so are its compactifications.

Lemma 3.19 *Let $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$. Then the following conditions are equivalent:*

(a) *$\mathcal{C}$ is bounded,*
(b) *$\mathcal{C}$ is bounded on $\Gamma \times \{x\}$ for some $x \in X$,*
(c) *$\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{R})$.*

Further, if $p\colon \mathbb{R} \to b\mathbb{R}$ is the Bohr compactification of $\mathbb{R}$ (see Sect. 2.5) then the following conditions are equivalent:

(d) *$p\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(b\mathbb{R})$,*
(e) *$\mathcal{C}^{\perp} = \mathbb{R}^*$.*

Finally, if X is metrizable then all the conditions listed above are mutually equivalent and equivalent to

(f) *$\mathcal{C}^{\times} = \mathbb{R}^*$.*

Proof The equivalence of (a)–(c) is well known, see [23, Theorem 14.11, p. 135] or [41, Proposition 2.1]. By Lemma 3.4, (d) is equivalent to $(p\mathcal{C})^{\perp} = (b\mathbb{R})^*$. Since $(p\mathcal{C})^{\perp} = (p^*)^{-1}(\mathcal{C}^{\perp})$ according to Lemma 3.8 and $p^*\colon (b\mathbb{R})^* \to \mathbb{R}^*$ is an isomorphism of groups, this translates into (e). Implication (e)⇒(f) is clear and so is (c)⇒(d). Finally, if X is metrizable and $\mathcal{C}^{\times} = \mathbb{R}^*$ then $\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{R})$ by Corollary 3.18, so (c) follows from (f). □

Remark 3.20 The metrizability assumption on X is essential for the equivalence of, say, (c) and (e). For if $\mathcal{F}$ is the universal minimal $\mathbb{R}$-flow and $\mathcal{C} = \mathcal{Q}_{\mathrm{id}}$ is the quasicoboundary induced by the identical morphism $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ then $\mathcal{C} \notin \mathbf{B}_{\mathcal{F}}(\mathbb{R})$ since $\mathcal{C}$ is unbounded, but $\mathcal{C}^{\perp} = \mathbb{R}^*$ by our Lemma 5.1.

The next lemma follows by the argument of [19, Lemma 1.2]. We will use it in our proofs of Theorems 1.1 and 1.2.

Lemma 3.21 *Let $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism of minimal flows. Then the morphism $h_{\mathbb{R}}\colon \mathbf{Z}_{\mathcal{F}}(\mathbb{R}) \to \mathbf{Z}_{\mathcal{G}}(\mathbb{R})$ defined by (2.13) fulfills*

$$h_{\mathbb{R}}^{-1}(\mathbf{B}_{\mathcal{G}}(\mathbb{R})) = \mathbf{B}_{\mathcal{F}}(\mathbb{R}).$$

Consequently, the induced morphism $h_{\mathbb{R}}\colon \mathbf{H}_{\mathcal{F}}(\mathbb{R}) \to \mathbf{H}_{\mathcal{G}}(\mathbb{R})$ is a monomorphism.

Proof Inclusion "$\supseteq$" is clear. To verify "$\subseteq$", fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ and write $\mathcal{D} = h_{\mathbb{R}}(\mathcal{C})$. Recall that $\mathcal{D}(\gamma, y) = \mathcal{C}(\gamma, h(y))$ for every $\gamma \in \Gamma$ and each y from the phase space Y of $\mathcal{G}$. Now assume that $\mathcal{D} \in \mathbf{B}_{\mathcal{G}}(\mathbb{R})$. Then $\mathcal{D}$ is bounded, hence by surjectivity of h, the same is true for $\mathcal{C}$. Thus $\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(\mathbb{R})$ by Lemma 3.19. □

4 Locally contractible groups

The aim of this section is a deeper analysis of various notions considered so far, under the additional assumption that the acting group Γ of $\mathcal{F}$ is connected and locally contractible. This is accomplished by changing the acting group Γ of $\mathcal{F}$ to its universal cover $\widetilde{\Gamma}$, thus obtaining a minimal $\widetilde{\Gamma}$-flow $\widetilde{\mathcal{F}}$, and by doing the same with cocycles over $\mathcal{F}$ by means of the morphism $\mathrm{uc}_G\colon \mathbf{Z}_{\mathcal{F}}(G) \to \mathbf{Z}_{\widetilde{\mathcal{F}}}(G)$ from Definition 4.1 below. Thus we show that

- the group $\mathbf{A}_{\mathcal{F}}$ from Definition 2.6 increases with $\mathcal{F}$ (Lemma 4.7),
- the group $\pi^1(\mathcal{F})$ from Definition 3.10 is expressible in the language of fundamental group (Proposition 4.8),
- $\pi^1(\mathcal{F})$ is a direct summand in $\pi^1(X)$ when the fundamental group of Γ is finitely generated (Corollary 4.9).

In this way we make important steps towards our proofs of Theorem 1.1 in Sect. 4.4 and Theorem 1.2 in Sect. 5.1.

4.1 The universal cover

Now we fix some notation, to be used throughout the whole section. Let Γ be connected and locally contractible. By [46, Theorem 82.1, p. 495], it has a universal cover $p\colon \widetilde{\Gamma} \to \Gamma$. Recall that $\widetilde{\Gamma}$ carries the structure of a topological group in such a way that p is a morphism of groups [46, p. 483]. Consider the flow

$$\widetilde{\mathcal{F}}\colon \widetilde{\Gamma} \curvearrowright X, \quad \widetilde{\mathcal{F}} := \mathcal{F}(p \times \mathrm{id}_X). \tag{4.1}$$

Since $\mathcal{F}, \widetilde{\mathcal{F}}$ share orbits and $\mathcal{F}$ is minimal, $\widetilde{\mathcal{F}}$ is also minimal.

Definition 4.1 Let Γ be connected and locally contractible, and let G be an abelian topological group. We define the map

$$\mathrm{uc}_G\colon \mathbf{Z}_{\mathcal{F}}(G) \to \mathbf{Z}_{\widetilde{\mathcal{F}}}(G), \quad \mathcal{C} \mapsto \mathcal{C}(p \times \mathrm{id}_X). \tag{4.2}$$

Lemma 4.2 *The map* uc_G *from Definition 4.1 is a monomorphism of groups with*

$$\mathrm{im}(\mathrm{uc}_G) = \{\widetilde{\mathcal{C}} \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(G) \colon \widetilde{\mathcal{C}}(\ker(p) \times X) = \{0\}\}. \tag{4.3}$$

Proof It follows by a standard verification that $\mathrm{uc}_G(\mathcal{C}) \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(G)$ for every $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, so uc_G is well defined. An elementary argument also shows that uc_G is a morphism of groups. Moreover, uc_G is a monomorphism since p is an epimorphism.

Now we verify (4.3). Inclusion "$\subseteq$" is clear. To verify inclusion "$\supseteq$", fix $\widetilde{\mathcal{C}} \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(G)$ and assume that $\widetilde{\mathcal{C}}(\ker(p) \times X) = \{0\}$. We show that there is a continuous map $\mathcal{C}\colon \Gamma \times X \to G$ with $\widetilde{\mathcal{C}} = \mathcal{C}(p \times \mathrm{id}_X)$. Since $p \times \mathrm{id}_X$ is open, hence a quotient map, it suffices to verify that $\widetilde{\mathcal{C}}$ identifies the pairs of points identified by $p \times \mathrm{id}_X$. Notice that such pairs have the form (α, x), $(\alpha\beta, x)$, where $\alpha \in \widetilde{\Gamma}$, $\beta \in \ker(p)$ and $x \in X$. Then $p(\beta) = 1$, hence by the cocycle identity (2.3),

$$\widetilde{\mathcal{C}}(\alpha\beta, x) = \widetilde{\mathcal{C}}(\alpha, \widetilde{\mathcal{F}}(\beta, x)) + \widetilde{\mathcal{C}}(\beta, x) = \widetilde{\mathcal{C}}(\alpha, \mathcal{F}(p(\beta), x)) + 0 = \widetilde{\mathcal{C}}(\alpha, x).$$

It is now easily checked that $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, hence $\widetilde{\mathcal{C}} = \mathrm{uc}_G(\mathcal{C}) \in \mathrm{im}(\mathrm{uc}_G)$. □

4.2 Auxiliary lemmas

This section contains some technical results, to be used later in the proofs of our main theorems. We begin by relating the morphism uc_G from (4.2) to the notions studied in the previous sections.

Lemma 4.3 *If* Γ *is connected and locally contractible then*

$$\mathrm{Exp}_{\widetilde{\mathcal{F}}}\mathrm{uc}_{\mathbb{R}} = \mathrm{uc}_{\mathbb{T}}\mathrm{Exp}_{\mathcal{F}}$$

(see Definition 2.3 and Fig. 3).

Fig. 3 The universal cover and the induced exponential

$$\begin{tikzcd}
\mathbf{Z}_{\mathcal{F}}(\mathbb{R}) \arrow[r, "\mathrm{uc}_{\mathbb{R}}"] \arrow[d, "\mathrm{Exp}_{\mathcal{F}}"'] & \mathbf{Z}_{\widetilde{\mathcal{F}}}(\mathbb{R}) \arrow[d, "\mathrm{Exp}_{\widetilde{\mathcal{F}}}"] \\
\mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \arrow[r, "\mathrm{uc}_{\mathbb{T}}"] & \mathbf{Z}_{\widetilde{\mathcal{F}}}(\mathbb{T})
\end{tikzcd}$$

Fig. 4 The universal cover and morphisms of cocycle groups

$$\begin{tikzcd}
\mathbf{Z}_{\mathcal{F}}(G) \arrow[r, "h_G"] \arrow[d, "\mathrm{uc}_G"'] & \mathbf{Z}_{\mathcal{G}}(G) \arrow[d, "\mathrm{uc}_G"] \\
\mathbf{Z}_{\widetilde{\mathcal{F}}}(G) \arrow[r, "\widetilde{h}_G"] & \mathbf{Z}_{\widetilde{\mathcal{G}}}(G)
\end{tikzcd}$$

Proof Fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$. Then

$$\begin{aligned}
\mathrm{Exp}_{\widetilde{\mathcal{F}}}\mathrm{uc}_{\mathbb{R}}(\mathcal{C}) &= \mathrm{Exp}_{\widetilde{\mathcal{F}}}\big(\mathcal{C}(p \times \mathrm{id}_X)\big) = \exp \mathcal{C}(p \times \mathrm{id}_X) = \mathrm{Exp}_{\mathcal{F}}(\mathcal{C})(p \times \mathrm{id}_X) \\
&= \mathrm{uc}_{\mathbb{T}}\mathrm{Exp}_{\mathcal{F}}(\mathcal{C}).
\end{aligned}$$

□

A morphism of minimal Γ-flows $h\colon \mathcal{G} \to \mathcal{F}$ may also be viewed as a morphism $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{F}}$. To avoid confusion, we denote the latter by

$$\widetilde{h}\colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{F}}. \tag{4.4}$$

The phase spaces of $\mathcal{F}$ and $\mathcal{G}$ will be denoted by X and Y, respectively.

Lemma 4.4 *Let Γ be connected and locally contractible, $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism of minimal Γ-flows and G be an abelian topological group. Then*

$$\mathrm{uc}_G h_G = \widetilde{h}_G \mathrm{uc}_G$$

(see (2.13) and Fig. 4).

Proof Fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. Then

$$\begin{aligned}
\mathrm{uc}_G h_G(\mathcal{C}) &= \mathrm{uc}_G\big(\mathcal{C}(\mathrm{id}_\Gamma \times h)\big) = \mathcal{C}(\mathrm{id}_\Gamma \times h)(p \times \mathrm{id}_Y) = \mathcal{C}(p \times h) = \mathcal{C}(p \times \mathrm{id}_X)(\mathrm{id}_{\widetilde{\Gamma}} \times h) \\
&= \mathrm{uc}_G(\mathcal{C})(\mathrm{id}_{\widetilde{\Gamma}} \times h) = \widetilde{h}_G \mathrm{uc}_G(\mathcal{C}).
\end{aligned}$$

□

Lemma 4.5 *Let Γ be connected and locally contractible and let $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism of minimal Γ-flows. Then*

$$\widetilde{h}_{\mathbb{R}}^{-1}(\mathrm{im}(\mathrm{uc}_{\mathbb{R}})) = \mathrm{im}(\mathrm{uc}_{\mathbb{R}}) \tag{4.5}$$

(see Fig. 4 with $G = \mathbb{R}$).

Proof Inclusion "$\supseteq$" follows from the commutativity of the diagram in Fig. 4. To verify inclusion "$\subseteq$", fix $\widetilde{\mathcal{C}} \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(\mathbb{R})$ and $\mathcal{D} \in \mathbf{Z}_{\mathcal{G}}(\mathbb{R})$ with $\widetilde{h}_{\mathbb{R}}(\widetilde{\mathcal{C}}) = \mathrm{uc}_{\mathbb{R}}(\mathcal{D})$. By Lemma 4.2, to show that $\widetilde{\mathcal{C}} \in \mathrm{im}(\mathrm{uc}_{\mathbb{R}})$, we need only verify that $\widetilde{\mathcal{C}}$ vanishes on $\ker(p) \times X$. So fix $\beta \in \ker(p)$, $x \in X$ and $y \in Y$ with $h(y) = x$. Then

$$\widetilde{\mathcal{C}}(\beta, x) = \widetilde{\mathcal{C}}(\beta, h(y)) = \widetilde{h}_{\mathbb{R}}(\widetilde{\mathcal{C}})(\beta, y) = \mathrm{uc}_{\mathbb{R}}(\mathcal{D})(\beta, y) = \mathcal{D}(p(\beta), y) = \mathcal{D}(1, y) = 0.$$

□

The following lemma will be used in our proof of Theorem 1.1 and it is also the main technical tool in proving Lemma 4.7 below, which will be used in our proof of Theorem 1.2.

Lemma 4.6 *Let Γ be connected and locally contractible and $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism of minimal Γ-flows. Then*

$$h_{\mathbb{T}}^{-1}(\mathrm{im}(\mathrm{Exp}_{\mathcal{G}})) = \mathrm{im}(\mathrm{Exp}_{\mathcal{F}}) \tag{4.6}$$

(see Fig. 1).

Proof Inclusion "$\supseteq$" is clear. When verifying inclusion "$\subseteq$", the reader might find it useful to draw the respective steps in a three-dimensional commutative diagram.

Fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ and assume that $h_{\mathbb{T}}(\mathcal{C}) = \mathrm{Exp}_{\mathcal{G}}(\mathcal{D})$ for some $\mathcal{D} \in \mathbf{Z}_{\mathcal{G}}(\mathbb{R})$. By Lemmas 4.4 and 4.3,

$$\widetilde{h}_{\mathbb{T}}\mathrm{uc}_{\mathbb{T}}(\mathcal{C}) = \mathrm{uc}_{\mathbb{T}}h_{\mathbb{T}}(\mathcal{C}) = \mathrm{uc}_{\mathbb{T}}\mathrm{Exp}_{\mathcal{G}}(\mathcal{D}) = \mathrm{Exp}_{\widetilde{\mathcal{G}}}\mathrm{uc}_{\mathbb{R}}(\mathcal{D}). \tag{4.7}$$

Since $\widetilde{\Gamma}$ is simply connected, we may use [12, Theorem 3.38, p. 84] to find $\widetilde{\mathcal{E}} \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(\mathbb{R})$ with $\mathrm{uc}_{\mathbb{T}}(\mathcal{C}) = \mathrm{Exp}_{\widetilde{\mathcal{F}}}(\widetilde{\mathcal{E}})$. Then it follows from Lemma 2.7 and (4.7) that

$$\mathrm{Exp}_{\widetilde{\mathcal{G}}}\widetilde{h}_{\mathbb{R}}(\widetilde{\mathcal{E}}) = \widetilde{h}_{\mathbb{T}}\mathrm{Exp}_{\widetilde{\mathcal{F}}}(\widetilde{\mathcal{E}}) = \widetilde{h}_{\mathbb{T}}\mathrm{uc}_{\mathbb{T}}(\mathcal{C}) = \mathrm{Exp}_{\widetilde{\mathcal{G}}}\mathrm{uc}_{\mathbb{R}}(\mathcal{D}).$$

Since $\mathrm{Exp}_{\widetilde{\mathcal{G}}}$ is a monomorphism by Lemma 2.4, we infer that $\widetilde{h}_{\mathbb{R}}(\widetilde{\mathcal{E}}) = \mathrm{uc}_{\mathbb{R}}(\mathcal{D})$. In view of Lemma 4.5, this yields $\mathcal{E} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $\widetilde{\mathcal{E}} = \mathrm{uc}_{\mathbb{R}}(\mathcal{E})$. Then by Lemma 4.4,

$$\mathrm{uc}_{\mathbb{R}}(\mathcal{D}) = \widetilde{h}_{\mathbb{R}}(\widetilde{\mathcal{E}}) = \widetilde{h}_{\mathbb{R}}\mathrm{uc}_{\mathbb{R}}(\mathcal{E}) = \mathrm{uc}_{\mathbb{R}}h_{\mathbb{R}}(\mathcal{E}),$$

and since $\mathrm{uc}_{\mathbb{R}}$ is a monomorphism on the account of Lemma 4.2, $\mathcal{D} = h_{\mathbb{R}}(\mathcal{E})$. Thus by Lemma 2.7,

$$h_{\mathbb{T}}(\mathcal{C}) = \mathrm{Exp}_{\mathcal{G}}(\mathcal{D}) = \mathrm{Exp}_{\mathcal{G}}h_{\mathbb{R}}(\mathcal{E}) = h_{\mathbb{T}}\mathrm{Exp}_{\mathcal{F}}(\mathcal{E}),$$

and since $h_{\mathbb{T}}$ is a monomorphism (see Sect. 2.15), $\mathcal{C} = \mathrm{Exp}_{\mathcal{F}}(\mathcal{E}) \in \mathrm{im}(\mathrm{Exp}_{\mathcal{F}})$ indeed. □

Now we show that $\mathbf{A}_{\mathcal{F}}$ is increasing in $\mathcal{F}$. This fact will be used when verifying (1.3) and then in turn when determining A_{Γ} from Theorem 1.2 in concrete situations of Sect. 5.

Lemma 4.7 *Let Γ be connected and locally contractible and $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism of minimal Γ-flows. Then $h_{\mathbb{T}}\colon \mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \to \mathbf{Z}_{\mathcal{G}}(\mathbb{T})$ induces a monomorphism*

$$\mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\,\mathrm{im}(\mathrm{Exp}_{\mathcal{F}}) \to \mathbf{Z}_{\mathcal{G}}(\mathbb{T})/\,\mathrm{im}(\mathrm{Exp}_{\mathcal{G}})$$

and $\mathbf{A}_{\mathcal{F}}$ is isomorphic to a subgroup of $\mathbf{A}_{\mathcal{G}}$ (see Definition 2.6).

Proof The first claim follows from Lemma 4.6. The second claim follows from the first claim and Lemma 2.5. □

4.3 Dynamical invariants via the fundamental group

The purpose of this section is to relate the groups $\pi^1(\mathcal{F})$ and $\mathbf{A}_{\mathcal{F}}$ from (3.7) and (2.17) to the fundamental group and induced morphisms. This will enable us to employ the fundamental group of Γ into the study of algebraic topology of its universal minimal flow.

Recall the transition map

$$\mathcal{F}_z\colon \Gamma \to X, \quad \gamma \mapsto \mathcal{F}(\gamma, z),$$

$$\begin{tikzcd}
\widetilde{\Gamma} \times X \arrow[r, "\widetilde{\mathcal{C}}"] \arrow[d, "p\times\mathrm{id}"'] & \mathbb{R} \arrow[d, "\exp"] \\
\Gamma \times X \arrow[r, "\mathcal{C}"] \arrow[ur, dashed, "\mathcal{D}"] & \mathbb{T}
\end{tikzcd}$$

Fig. 5 Lifts and projections of cocycles

where z is the base point of X, and the notation T_γ ($\gamma \in \Gamma$) for the acting homeomorphisms of $\mathcal{F}$. Similarly, given an abelian topological group G and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$, set

$$\mathcal{C}_z \colon \Gamma \to G, \quad \gamma \mapsto \mathcal{C}(\gamma, z). \tag{4.8}$$

The morphism of fundamental groups induced by a continuous base point preserving map f will be denoted by f_*.

Proposition 4.8 *Let* Γ *be connected and locally contractible. Then*

$$\mathrm{im}(\mathrm{Exp}_{\mathcal{F}}) = \{\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \colon (\mathcal{C}_z)_* = 0\} \tag{4.9}$$

and

$$\pi^1(\mathcal{F}) = \{\xi \in \pi^1(X) \colon \xi_*(\mathcal{F}_z)_* = 0\}. \tag{4.10}$$

Moreover, $\mathbf{A}_{\mathcal{F}}$ *is isomorphic to a subgroup of* $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$ *and* $\pi^1(X)/\pi^1(\mathcal{F})$ *is isomorphic to a subgroup of* $\mathrm{Hom}(\mathrm{im}((\mathcal{F}_z)_*), \mathbb{Z})$.

Proof We divide the proof into four steps.

STEP 1. We verify (4.9).

Fix $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$. If $\mathcal{C} = \mathrm{Exp}_{\mathcal{F}}(\mathcal{D})$ for some $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ then $\mathcal{C}_z = \exp \mathcal{D}_z$, hence $(\mathcal{C}_z)_* = \exp_*(\mathcal{D}_z)_* = 0$ by simple connectedness of $\mathbb{R}$. Conversely, let $(\mathcal{C}_z)_* = 0$. We proceed as pictured in Fig. 5. Since $\widetilde{\Gamma}$ is simply connected, we may use [12, Theorem 3.38, p. 84] to find $\widetilde{\mathcal{C}} \in \mathbf{Z}_{\widetilde{\mathcal{F}}}(\mathbb{R})$ with $\mathrm{uc}_{\mathbb{T}}(\mathcal{C}) = \mathrm{Exp}_{\widetilde{\mathcal{F}}}(\widetilde{\mathcal{C}})$. We claim that $\widetilde{\mathcal{C}}$ vanishes on $\ker(p) \times X$. To verify this, fix $\delta \in \ker(p)$ and $\gamma \in \Gamma$. Choose a path f in $\widetilde{\Gamma}$ from 1 to δ and a path g in X from z to $T_\gamma z$. Clearly, pf is a loop in Γ based at 1. In order to simplify notation, the following expressions involving t are understood as functions of variable t and $\simeq$ denotes the relation of homotopy of paths. Thus we have, in the notation of Sect. 2.6,

$$\begin{aligned}\exp \widetilde{\mathcal{C}}(f(t), g(t)) &= \mathcal{C}(pf(t), g(t)) \simeq \mathcal{C}(pf(t) * 1, z * g(t)) = \mathcal{C}\Big[(pf(t), z) * (1, g(t))\Big] \\ &= \mathcal{C}(pf(t), z) * \mathcal{C}(1, g(t)) = \mathcal{C}_z pf(t) * 1 \simeq \mathcal{C}_z pf(t) \simeq 1.\end{aligned}$$

Since $\widetilde{\mathcal{C}}(f(0), g(0)) = \widetilde{\mathcal{C}}(1, z) = 0$, we infer that $\widetilde{\mathcal{C}}(f(t), g(t))$ is a loop based at 0 [46, Theorem 54.3, p. 344], hence $\widetilde{\mathcal{C}}(\delta, T_\gamma z) = \widetilde{\mathcal{C}}(f(1), g(1)) = 0$. Thus $\widetilde{\mathcal{C}}(\ker(p) \times X) = \{0\}$, hence by Lemma 4.2, $\widetilde{\mathcal{C}} = \mathrm{uc}_{\mathbb{R}}(\mathcal{D})$ for some $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$. In view of Lemma 4.3, it follows that

$$\mathrm{uc}_{\mathbb{T}} \mathrm{Exp}_{\mathcal{F}}(\mathcal{D}) = \mathrm{Exp}_{\widetilde{\mathcal{F}}} \mathrm{uc}_{\mathbb{R}}(\mathcal{D}) = \mathrm{Exp}_{\widetilde{\mathcal{F}}}(\widetilde{\mathcal{C}}) = \mathrm{uc}_{\mathbb{T}}(\mathcal{C}),$$

and since $\mathrm{uc}_{\mathbb{T}}$ is a monomorphism by Lemma 4.2, $\mathcal{C} = \mathrm{Exp}_{\mathcal{F}}(\mathcal{D}) \in \mathrm{im}(\mathrm{Exp}_{\mathcal{F}})$ indeed.

STEP 2. We verify (4.10).

Fix a continuous base point preserving map $\xi \colon X \to \mathbb{T}$. By definition (3.7) of $\pi^1(\mathcal{F})$, $\xi \in \pi^1(\mathcal{F})$ if, and only if, $\mathrm{co}(\xi) \in \mathrm{im}(\mathrm{Exp}_{\mathcal{F}})$. In view of (4.9), this is equivalent to $(\mathrm{co}(\xi)_z)_* = 0$. Since $\mathrm{co}(\xi)_z = \xi \mathcal{F}_z$, the latter equality occurs precisely when $\xi_*(\mathcal{F}_z)_* = 0$.

STEP 3. We show that $\mathbf{A}_{\mathcal{F}}$ is isomorphic to a subgroup of $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$.

Since the group operation in $\pi_1(\mathbb{T})$ coincides with the pointwise product of loops, the assignment

$$\varphi\colon \mathbf{Z}_{\mathcal{F}}(\mathbb{T}) \to \operatorname{Hom}(\pi_1(\Gamma), \pi_1(\mathbb{T})), \quad \mathcal{C} \mapsto (\mathcal{C}_z)_*,$$

defines a morphism of groups. By Step 1, $\ker(\varphi) = \operatorname{im}(\operatorname{Exp}_{\mathcal{F}})$, hence φ gives rise to a monomorphism

$$\mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\operatorname{im}(\operatorname{Exp}_{\mathcal{F}}) \to \operatorname{Hom}(\pi_1(\Gamma), \pi_1(\mathbb{T})).$$

Now it suffices to use the isomorphism $\mathbf{A}_{\mathcal{F}} \cong \mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\operatorname{im}(\operatorname{Exp}_{\mathcal{F}})$ from Lemma 2.5 and the usual isomorphism $\pi_1(\mathbb{T}) \cong \mathbb{Z}$.

STEP 4. We show that $\pi^1(X)/\pi^1(\mathcal{F})$ is isomorphic to a subgroup of $\operatorname{Hom}(\operatorname{im}((\mathcal{F}_z)_*), \mathbb{Z})$.

It follows from our discussion in Sect. 2.10 that the assignment

$$\pi^1(X) \to \operatorname{Hom}(\operatorname{im}((\mathcal{F}_z)_*), \pi_1(\mathbb{T})), \quad \xi \mapsto \xi_*|_{\operatorname{im}((\mathcal{F}_z)_*)},$$

defines a morphism of groups, whose kernel is $\pi^1(\mathcal{F})$ by (4.10). Thus we have a monomorphism

$$\pi^1(X)/\pi^1(\mathcal{F}) \to \operatorname{Hom}(\operatorname{im}((\mathcal{F}_z)_*), \pi_1(\mathbb{T})).$$

□

Corollary 4.9 *If Γ is connected, locally contractible and with a finitely generated fundamental group then there is a direct sum*

$$\pi^1(X) = \pi^1(\mathcal{F}) \oplus A, \tag{4.11}$$

where A is a free abelian group with a finite rank

$$\mathrm{r}(A) \leq \mathrm{r}(\operatorname{im}((\mathcal{F}_z)_*)) \leq \mathrm{r}(\pi_1(\Gamma)). \tag{4.12}$$

Proof First notice that since $\pi_1(\Gamma)$ is abelian and finitely generated, the same is true for $\operatorname{im}((\mathcal{F}_z)_*)$. It follows that $\operatorname{Hom}(\operatorname{im}((\mathcal{F}_z)_*), \mathbb{Z})$ is free abelian with the finite rank $\mathrm{r}(\operatorname{im}((\mathcal{F}_z)_*))$, hence by the last assertion of Proposition 4.8, $\pi^1(X)/\pi^1(\mathcal{F})$ is free abelian with rank not exceeding $\mathrm{r}(\operatorname{im}((\mathcal{F}_z)_*))$. Since free abelian groups are projective, we obtain (4.11) with A isomorphic to $\pi^1(X)/\pi^1(\mathcal{F})$, thus fulfilling (4.12). □

4.4 Morphisms of minimal flows

In this section we prove Theorem 1.1 and derive from it two consequences, namely Corollary 4.10 and Proposition 4.12. Our proof of Theorem 1.1 can be viewed as analogous to the complexification argument from the classical mathematical analysis. Methods of complex analysis, though unavailable in $\mathbb{R}$, can nevertheless be used to prove real theorems due to the natural inclusion $\mathbb{R} \subseteq \mathbb{C}$. Similarly, by assigning to each continuous map $\xi\colon X \to \mathbb{T}$ the corresponding coboundary $\operatorname{co}(\xi)$, we identify the group $C_z(X, \mathbb{T})$ of continuous base point preserving maps $X \to \mathbb{T}$ with the group of coboundaries $\mathbf{B}_{\mathcal{F}}(\mathbb{T}) \subseteq \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ and then use methods applicable to cocycles but not to coboundaries alone. As before, we write X and Y for the phase spaces of minimal flows $\mathcal{F}$ and $\mathcal{G}$, respectively.

Proof of Theorem 1.1 Let $\xi\colon X \to \mathbb{T}$ be continuous and base point preserving and assume that $h^\flat(\xi) = 0$ in $\pi^1(Y)$. Then $\xi h = \exp\vartheta$ for some continuous base point preserving map $\vartheta\colon Y \to \mathbb{R}$. Consequently, by (2.14),

$$h_{\mathbb{T}}(\operatorname{co}(\xi)) = \operatorname{co}(\xi h) = \operatorname{co}(\exp\vartheta) = \operatorname{Exp}_{\mathcal{G}}(\operatorname{co}(\vartheta)),$$

hence $\mathrm{co}(\xi) \in h_{\mathbb{T}}^{-1}(\mathrm{im}(\mathrm{Exp}_{\mathcal{G}}))$. By Lemma 4.6, there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $\mathrm{co}(\xi) = \mathrm{Exp}_{\mathcal{F}}(\mathcal{C})$. In view of Lemma 2.7, it follows that

$$\mathrm{Exp}_{\mathcal{G}} h_{\mathbb{R}}(\mathcal{C}) = h_{\mathbb{T}} \mathrm{Exp}_{\mathcal{F}}(\mathcal{C}) = h_{\mathbb{T}}(\mathrm{co}(\xi)) = \mathrm{Exp}_{\mathcal{G}}(\mathrm{co}(\vartheta)).$$

Since $\mathrm{Exp}_{\mathcal{G}}$ is a monomorphism according to Lemma 2.4, we infer that $h_{\mathbb{R}}(\mathcal{C}) = \mathrm{co}(\vartheta)$. Thus $\mathcal{C} \in h_{\mathbb{R}}^{-1}(\mathbf{B}_{\mathcal{G}}(\mathbb{R}))$ and it follows from Lemma 3.21 that $\mathcal{C} = \mathrm{co}(\eta)$ for some continuous base point preserving map $\eta\colon X \to \mathbb{R}$. Then

$$\mathrm{co}(\xi) = \mathrm{Exp}_{\mathcal{F}}(\mathrm{co}(\eta)) = \mathrm{co}(\exp \eta),$$

whence it follows that $\xi = \exp \eta$. Thus $\xi = 0$ in $\pi^1(X)$, as was to be shown. □

The following result applies to flows on manifolds. (Recall that $H_1^w(X)$ and $H_1^w(Y)$ are the first weak homology groups of X and Y, respectively.)

Corollary 4.10 *Let Γ be connected and locally contractible, $\mathcal{F}$ and $\mathcal{G}$ be minimal Γ-flows defined on manifolds and $h\colon \mathcal{G} \to \mathcal{F}$ be a morphism. Then the induced morphism $h^\sharp\colon H_1^w(Y) \to H_1^w(X)$ has maximal rank in the sense that*

$$\mathrm{r}(\mathrm{im}(h^\sharp)) = \mathrm{r}(H_1^w(X)).$$

Consequently,

$$\mathrm{r}(H_1^w(Y)) \geq \mathrm{r}(H_1^w(X)).$$

Proof In the notation of Sect. 2.10, we have

$$\sigma_Y h^\flat \sigma_X^{-1}(\varphi) = \varphi h^\sharp$$

for every $\varphi \in \mathrm{Hom}(H_1^w(X), H_1^w(\mathbb{T}))$ by Lemma 2.1. Since $h^\flat$ is a monomorphism by Theorem 1.1 and σ_X, σ_Y are isomorphisms, the assignment $\varphi \mapsto \varphi h^\sharp$ defines a monomorphism. By the structure theorem for finitely generated abelian groups and their subgroups, this is possible only when $\mathrm{im}(h^\sharp)$ and $H_1^w(X)$ have the same rank. The second statement of the corollary is clear. □

The morphism $h^\sharp$ may fail to be an epimorphism and unlike in the special situation of [3, Theorem 5] (see Sect. 1.1), the cokernel of $h^\flat$ is not torsion-free in general.

Example 4.11 Fix integers $n \geq 1$ and $d \geq 2$. Let $\mathcal{G}$ be a minimal linear $\mathbb{R}$-flow on the torus $\mathbb{T}^n$ and $\mathcal{F}$ be the flow obtained from $\mathcal{G}$ by increasing the speed d times. The d-endomorphism h of $\mathbb{T}^n$ is a morphism $\mathcal{G} \to \mathcal{F}$ and it induces the d-endomorphism $h^\sharp$ of $H_1^w(\mathbb{T}^n)$, which is not an epimorphism. Moreover, the induced morphism $h^\flat\colon \pi^1(\mathbb{T}^n) \to \pi^1(\mathbb{T}^n)$ may be identified with the d-endomorphism of $\mathbb{Z}^n$, hence

$$\mathrm{coker}(h^\flat) = \pi^1(\mathbb{T}^n)/\,\mathrm{im}(h^\flat) \cong \mathbb{Z}^n/d\mathbb{Z}^n \cong (\mathbb{Z}_d)^n.$$

Recall that when the acting group of a minimal equicontinuous flow $\mathcal{E}\colon \Gamma \curvearrowright Z$ is abelian then Z carries the structure of a compact abelian group in which Γ compactifies [55, Theorem 3.38, p. 316].

Proposition 4.12 *Let Γ be connected, locally contractible and abelian, $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow and $\mathcal{E}\colon \Gamma \curvearrowright Z$ be its maximal equicontinuous factor. Then Z is topologically isomorphic to a quotient group of the Pontryagin dual $\pi^1(X)^*$ of $\pi^1(X)$. If, in addition, X is a manifold then Z is a torus of dimension*

$$0 \leq \dim(Z) \leq \mathrm{r}(H_1^w(X)). \tag{4.13}$$

Proof Denote the morphism $\mathcal{F} \to \mathcal{E}$ by h. By Theorem 1.1, $h^\flat \colon \pi^1(Z) \to \pi^1(X)$ is a monomorphism. Thus $Z^* \cong \pi^1(Z)$ is isomorphic to a subgroup of $\pi^1(X)$, hence $Z \cong Z^{**}$ is topologically isomorphic to a quotient group of $\pi^1(X)^*$. Further, if X is a manifold then $\pi^1(X) \cong H_1^w(X)$ is a free abelian group of a finite rank, so Z^* is also free abelian and with rank not exceeding $\mathrm{r}(H_1^w(X))$. Thus Z is a torus fulfilling (4.13). □

5 Universal minimal flows

This last section is devoted to the topology of universal minimal flows. In its first part we employ the groups $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$, $\pi^1(\mathcal{U}(\Gamma))$ and $\mathbf{A}_{\mathcal{U}(\Gamma)}$ from (2.5), (3.7) and (2.17), associated now to the universal minimal flow $\mathcal{U}(\Gamma)$ of Γ, to prove Theorem 1.2 for connected, locally contractible groups having a finitely generated fundamental group, and its Corollary 5.8 for simply connected groups Γ. Subsequently, we illustrate how Theorem 1.2 may be used in concrete situations, by computing $\pi^1(M(\Gamma))$ for certain connected Lie groups and for identity components of homeomorphism groups of some compact connected surfaces. We conclude by presenting a proof of Theorem 1.3.

Keeping the notation used so far, we denote the universal minimal flow associated to a topological group Γ by $\mathcal{U}(\Gamma) \colon \Gamma \curvearrowright M(\Gamma)$. We assume that the space $M(\Gamma)$ is pointed with the base point z.

5.1 First cohomotopy groups of universal minimal flows

Now we focus our attention on algebraic topology of $M(\Gamma)$. We begin with a simple consequence of the coalescence of $\mathcal{U}(\Gamma)$.

Lemma 5.1 *We have* $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{T}) = 0$.

Proof Given $\mathcal{C} \in \mathbf{Z}_{\mathcal{U}(\Gamma)}(\mathbb{T})$, let $\mathcal{M}$ be the restriction of the group extension $\mathcal{U}(\Gamma)_{\mathcal{C}}$ defined by (2.2) onto one of its minimal sets. By universality of $\mathcal{U}(\Gamma)$, there is a morphism $q \colon \mathcal{U}(\Gamma) \to \mathcal{M}$. Moreover, the projection on the first coordinate is a morphism $\mathrm{pr} \colon \mathcal{M} \to \mathcal{U}(\Gamma)$. Thus $\mathrm{pr}\, q$ is an endomorphism of $\mathcal{U}(\Gamma)$ and since $\mathcal{U}(\Gamma)$ is coalescent, it is in fact an automorphism. It follows that pr is injective, hence $F(\mathcal{C}) = \{1\}$ (see Sect. 2.14) and so $\mathcal{C} \in \mathbf{B}_{\mathcal{U}(\Gamma)}(\mathbb{T})$ by Lemma 3.4. □

Proposition 5.2 *If* Γ *is connected then there is a direct sum*

$$\pi^1(M(\Gamma)) = \pi^1(\mathcal{U}(\Gamma)) \oplus A_\Gamma, \tag{5.1}$$

where

$$\pi^1(\mathcal{U}(\Gamma)) \cong \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \quad \text{and} \quad A_\Gamma \cong \mathbf{A}_{\mathcal{U}(\Gamma)}. \tag{5.2}$$

If Γ *is also locally contractible then* A_Γ *is isomorphic to a subgroup of* $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$.

Proof By Proposition 3.16 and Lemma 5.1, there is an isomorphism of groups

$$\psi \colon \pi^1(M(\Gamma)) \to \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \oplus \mathbf{A}_{\mathcal{U}(\Gamma)}$$

with $\pi^1(\mathcal{U}(\Gamma)) = \psi^{-1}(\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}))$, so it suffices to take $A_\Gamma = \psi^{-1}(\mathbf{A}_{\mathcal{U}(\Gamma)})$. The second claim of the proposition follows from Proposition 4.8. □

Remark 5.3 Notice the following special cases of Proposition 5.2.

(a) If $\Gamma = \mathcal{H}_+(\mathbb{I})$, the group of orientation preserving homeomorphisms of the unit compact interval, equipped with the compact-open topology, then $M(\Gamma)$ is a singleton by [47, Theorem 6.2], so $\pi^1(\mathcal{U}(\Gamma)) = A_\Gamma = 0$.
(b) If $\Gamma = \mathcal{H}_+(\mathbb{T})$ then $M(\Gamma) = \mathbb{T}$ by [47, Theorem 6.6], so $\pi^1(\mathcal{U}(\Gamma)) = 0$ and $A_\Gamma \cong \mathbb{Z}$.
(c) It may happen that $\pi^1(\mathcal{U}(\Gamma)) \neq 0$ and $A_\Gamma = 0$ (Example 5.9) or that $\pi^1(\mathcal{U}(\Gamma)) \neq 0$ and $A_\Gamma \neq 0$ (Example 5.18).

Thus we see that the two direct summands from (5.1) are, in a sense, independent.

The following corollary applies to metrizable universal minimal flows.

Corollary 5.4 *Let Γ be connected and locally contractible. If $M(\Gamma)$ is metrizable then $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) = 0$ and $\pi^1(M(\Gamma))$ is isomorphic to a subgroup of* $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$.

Remark 5.5 If $\pi_1(\Gamma)$ is finitely generated then $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z})$ is isomorphic to the torsion-free part of $\pi_1(\Gamma)$, so $\pi^1(M(\Gamma))$ is a free abelian group of rank at most $\mathrm{r}(\pi_1(\Gamma))$. In particular, when Γ is simply connected then $\pi^1(M(\Gamma)) = 0$.

Proof of Corollary 5.4 By the metrizability assumption, $\pi^1(M(\Gamma))$ is countable, hence so is $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ (see (5.2)). Being a real linear space, $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ is thus necessarily trivial. The second claim of the corollary now follows from Proposition 5.2. □

The following lemma should be compared with (4.10) from Proposition 4.8. (Recall that ξ_* is the morphism of fundamental groups induced by a continuous map ξ.)

Lemma 5.6 *If Γ is connected and locally contractible then*

$$\pi^1(\mathcal{U}(\Gamma)) = \{\xi \in \pi^1(M(\Gamma))\colon \xi_* = 0\}.$$

Proof Inclusion "$\supseteq$" follows from (4.10). To verify inclusion "$\subseteq$", fix a continuous base point preserving map $\xi\colon M(\Gamma) \to \mathbb{T}$, representing an element of $\pi^1(\mathcal{U}(\Gamma))$, and a loop f in $M(\Gamma)$ based at z. Being isomorphic to the divisible group $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ by Proposition 5.2, $\pi^1(\mathcal{U}(\Gamma))$ is also divisible, so for every $d \in \mathbb{N}$ there exist continuous base point preserving maps $\zeta\colon M(\Gamma) \to \mathbb{T}$ and $\eta\colon M(\Gamma) \to \mathbb{R}$ with $\xi = \zeta^d \cdot \exp \eta$. By letting $\vartheta = \zeta \cdot \exp(\eta/d)$, we have $\xi = \vartheta^d$ pointwise, hence $\xi f = (\vartheta f)^d$. Consequently, $\xi_*(f) = d \cdot \vartheta_*(f)$ in $\pi_1(\mathbb{T})$ and it follows that $\xi_*(f)$ is a divisible element of $\pi_1(\mathbb{T})$. Thus $\xi_*(f) = 0$. □

To prove Theorem 1.2 we shall need the following estimate.

Lemma 5.7 *If Γ is connected and separable then $\mathrm{r}(\pi^1(M(\Gamma))) \leq \mathfrak{c}$.*

Proof By assumptions $M(\Gamma)$ is connected and separable. Due to separability, the group of continuous maps $C(M(\Gamma), \mathbb{T})$ has cardinality at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, hence the same is true for its quotient group $\pi^1(M(\Gamma))$. Thus

$$\mathrm{r}(\pi^1(M(\Gamma))) \leq \mathrm{card}(\pi^1(M(\Gamma))) \leq \mathfrak{c}.$$

□

From now on we shall consider groups Γ having finitely generated fundamental groups $\pi_1(\Gamma)$. These are covered by our Theorem 1.2. (Recall that the symbol $\mathrm{Div}(A)$ denotes the group of all divisible elements of an abelian group A.)

Proof of Theorem 1.2 By Proposition 5.2, we have (1.1) with $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$, $A_\Gamma \cong \mathbf{A}_{\mathcal{U}(\Gamma)}$ and A_Γ isomorphic to a subgroup of $\operatorname{Hom}(\pi_1(\Gamma), \mathbb{Z})$. Since $\pi_1(\Gamma)$ is abelian and finitely generated, $\operatorname{Hom}(\pi_1(\Gamma), \mathbb{Z})$ is free abelian with rank $\mathrm{r}(\pi_1(\Gamma))$, so A_Γ is also free abelian with rank at most $\mathrm{r}(\pi_1(\Gamma))$. If $\mathcal{F}$ is a minimal Γ-flow then $\mathbf{A}_{\mathcal{F}}$ is isomorphic to a subgroup of $\mathbf{A}_{\mathcal{U}(\Gamma)}$ according to Lemma 4.7, whence follows the second equality in (1.3). Further, $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ is a real linear space, so $\pi^1(\mathcal{U}(\Gamma)) \neq 0$ implies $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$ on the account of Lemma 5.7. Moreover, since $\pi^1(\mathcal{U}(\Gamma))$ is divisible and A_Γ has no divisible elements, we have $\pi^1(\mathcal{U}(\Gamma)) = \operatorname{Div}(\pi^1(M(\Gamma)))$. Finally, the equivalence of (1) and (2) is clear, while (2) is equivalent to (3) on the account of Lemma 3.21. □

For simply connected groups Γ we have the following corollary.

Corollary 5.8 *If Γ is separable, connected, locally contractible and simply connected then the following conditions are equivalent:*

(a) $\pi^1(M(\Gamma)) \cong \mathbb{R}$,
(b) $\pi^1(M(\Gamma)) \neq 0$,
(c) $\pi^1(X) \neq 0$ *for some minimal flow* $\mathcal{F}\colon \Gamma \curvearrowright X$,
(d) $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \neq 0$,
(e) $\mathbf{H}_{\mathcal{F}}(\mathbb{R}) \neq 0$ *for some minimal flow* $\mathcal{F}\colon \Gamma \curvearrowright X$.

Proof Since $\pi_1(\Gamma) = 0$ by assumption, we have $A_\Gamma = 0$ by Proposition 5.2, hence $\pi^1(M(\Gamma)) = \pi^1(\mathcal{U}(\Gamma)) \cong \mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ according to (1.1) and (1.2). The equivalence of (a), (b), (d) and (e) thus follows from Theorem 1.2. Condition (b) is equivalent to (c) by Theorem 1.1. □

Now we apply Corollary 5.8 in concrete situations.

Example 5.9 Let a simply connected Lie group Γ admit a nonzero topological morphism $h\colon \Gamma \to \mathbb{R}$. Then the quasicoboundary $\mathcal{Q}_h \in \mathbf{Z}_{\mathcal{U}(\Gamma)}(\mathbb{R})$ (see Sect. 2.13) represents a nonzero element of $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R})$, so $\pi^1(M(\Gamma)) \cong \mathbb{R}$ by Corollary 5.8. This applies to the groups $\mathbb{R}^n$ ($n \geq 1$) or the Heisenberg groups $H_n(\mathbb{R})$ ($n \geq 3$).

Example 5.10 Let Γ be a nontrivial separable locally convex real linear topological space. By the Hahn-Banach theorem, Γ has a nonzero continuous linear functional, so $\mathbf{H}_{\mathcal{U}(\Gamma)}(\mathbb{R}) \neq 0$ similarly to Example 5.9. Since Γ is also contractible and locally contractible, Corollary 5.8 yields $\pi^1(M(\Gamma)) \cong \mathbb{R}$.

Remark 5.11 In connection with Example 5.9 notice that a group Γ fulfilling assumptions of Corollary 5.8 admits a nontrivial morphism $0 \neq h \in \operatorname{Hom}(\Gamma, \mathbb{R})$ precisely when it has a nontrivial character $1 \neq \chi \in \operatorname{Hom}(\Gamma, \mathbb{T})$. In such a case the equicontinuous flow $\mathcal{F}\colon \Gamma \curvearrowright \mathbb{T}$ induced by χ is minimal and since $\pi^1(\mathbb{T}) \neq 0$, we may employ condition (c) from Corollary 5.8 to conclude that $\pi^1(M(\Gamma)) \cong \mathbb{R}$.

Recall that $H_1^w(X)$ denotes the first weak homology group of X and $\mathcal{H}_0(X)$ is the identity component of $\mathcal{H}(X)$.

Proposition 5.12 *Let X be a compact connected manifold without boundary and with $H_1^w(X) \neq 0$. If Γ denotes the universal covering group of $\mathcal{H}_0(X)$ then $\pi^1(M(\Gamma)) \cong \mathbb{R}$.*

Proof The group Γ inherits local contractibility from $\mathcal{H}_0(X)$; we show that it is also separable. Let $q\colon \Gamma \to \mathcal{H}_0(X)$ be the covering morphism. Choose symmetric identity neighbourhoods

$W \subseteq \mathcal{H}_0(X)$ and $V \subseteq \Gamma$, with V being mapped homeomorphically onto W by q. By separability of $\mathcal{H}_0(X)$, W is separable, hence so is V. Due to connectedness, Γ is generated by V and the claim thus follows.

Now we prove the proposition. Being transitive, the evaluation flow $\mathcal{H}_0(X) \curvearrowright X$ from (2.11) gives rise via q to a transitive flow $\mathcal{F}\colon \Gamma \curvearrowright X$. Since $\pi^1(X) \cong H_1^w(X) \neq 0$ by assumption, we have $\pi^1(M(\Gamma)) \cong \mathbb{R}$ according to Corollary 5.8. □

Remark 5.13 Keeping the notation from Proposition 5.12, let $X = \mathbb{T}$. Then the universal minimal flow of $\mathcal{H}_0(\mathbb{T}) = \mathcal{H}_+(\mathbb{T})$ is metrizable by [47, Theorem 6.6], but the universal minimal flow of Γ fulfills $\pi^1(M(\Gamma)) \cong \mathbb{R}$, hence is nonmetrizable.

Given a compact connected Lie group G, consider the loopgroup ΩG of G, formed by the continuous base point preserving maps $f\colon \mathbb{T} \to G$. Recall that ΩG is a topological group with pointwise defined operations and the compact-open topology.

Proposition 5.14 *Let G be a compact connected Lie group and Γ be the identity component of ΩG. If the fundamental group $\pi_1(G)$ of G is infinite then $\pi^1(M(\Gamma)) \cong \mathbb{R}$.*

Proof First we show that Γ satisfies the four topological requirements from Corollary 5.8. Its connectedness is clear, separability well known and local contractibility easy to verify. By [31, p. 395] and [49, p. 29],

$$\pi_1(\Gamma) = \pi_1(\Omega G) \cong \pi_2(G) = 0,$$

so Γ is also simply connected.

We prove the proposition by verifying (c) from Corollary 5.8. The assignment

$$\mathcal{F}\colon \Gamma \times G \to G, \quad (f, x) \mapsto f(-1) \cdot x,$$

defines a Γ-flow on G. Since $\pi^1(G) \neq 0$ by assumption, we need only show that $\mathcal{F}$ is minimal. Given $x, y \in G$, fix a path p in G from 1 to yx^{-1} and let f be the element of ΩG representing the loop $p * \overline{p}$ (see Sect. 2.6). Clearly, $f \in \Gamma$ and $\mathcal{F}(f, x) = y$. Thus $\mathcal{F}$ is transitive, hence minimal. □

Remark 5.15 A compact connected Lie group has infinite fundamental group if, and only if, it is not semisimple [10, Corollary 4, p. 285]. This family of groups includes tori or unitary groups.

5.2 The computability of $\pi^1(M(\Gamma))$

Now we show how to compute $\pi^1(M(\Gamma))$ in concrete situations. Our interest in this section is in connected Lie groups Γ. We keep our notation from Theorem 1.2 and from Sects. 2.6, 2.7 and 2.8. For the notions of topological freeness and free cycles, see Sect. 2.8.

Proposition 5.16 *Let Γ be a connected Lie group and $\mathcal{F}$ be a minimal Γ-flow on a manifold X. Then the following statements hold:*

(i) *if $\mathcal{F}$ is topologically free then* $\mathrm{r}(\mathbf{A}_{\mathcal{F}}) = \mathrm{r}(A_\Gamma) = \mathrm{r}(\pi_1(\Gamma))$,
(ii) *if $\mathcal{F}$ has a free cycle then* $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$.

Proof We proceed in two steps.

Step 1. We verify statement (i).

Since $\mathrm{r}(\pi_1(\Gamma)) = \mathrm{r}(H_1^w(\Gamma)) = \mathrm{r}(H_1^w(\mathcal{F}))$ by assumption and $\mathrm{r}(\mathbf{A}_{\mathcal{F}}) \le \mathrm{r}(A_\Gamma) \le \mathrm{r}(\pi_1(\Gamma))$ by Theorem 1.2, it suffices to show that $\mathrm{r}(H_1^w(\mathcal{F})) \le \mathrm{r}(\mathbf{A}_{\mathcal{F}})$. Write $n = \mathrm{r}(H_1^w(\mathcal{F}))$ and assume that $n \neq 0$. By the structure theorem for finitely generated abelian groups and their subgroups, there exist morphisms $\varrho_i \colon H_1^w(X) \to H_1^w(\mathbb{T})$ $(i = 1, \dots, n)$, whose restrictions onto $H_1^w(\mathcal{F})$ are independent over $\mathbb{Z}$.

For every i, let $\xi_i \colon X \to \mathbb{T}$ be continuous, base point preserving and with $\xi_i^\sharp = \varrho_i$. We claim that the coboundaries $\mathrm{co}(\xi_i) \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ $(i = 1, \dots, n)$ correspond to independent elements of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\operatorname{im}(\mathrm{Exp})$; since $\mathbf{A}_{\mathcal{F}} \cong \mathbf{Z}_{\mathcal{F}}(\mathbb{T})/\operatorname{im}(\mathrm{Exp})$ by Lemma 2.5, this will justify the desired inequality $n \le \mathrm{r}(\mathbf{A}_{\mathcal{F}})$. Using the additive notation in both $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ and $C(X, \mathbb{T})$, assume that $k_1, \dots, k_n \in \mathbb{Z}$ and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ are such that $\sum_{i=1}^n k_i \,\mathrm{co}(\xi_i) = \exp \mathcal{C}$. Since $\mathbb{R}$ is simply connected, $(\exp \mathcal{C})^\sharp = \exp^\sharp \mathcal{C}^\sharp = 0$, hence by [12, Lemma 2.17, p. 55],

$$
\begin{aligned}
0 &= \Big(\sum_{i=1}^n k_i \,\mathrm{co}(\xi_i) \Big)^\sharp (H_1^w(\Gamma \times X)) = \mathrm{co} \Big(\sum_{i=1}^n k_i \xi_i \Big)^\sharp (H_1^w(\Gamma \times X)) \\
&= \Big(\sum_{i=1}^n k_i \xi_i \Big)^\sharp (H_1^w(\mathcal{F})) = \Big(\sum_{i=1}^n k_i \xi_i^\sharp \Big)(H_1^w(\mathcal{F})) = \Big(\sum_{i=1}^n k_i \varrho_i \Big)(H_1^w(\mathcal{F})).
\end{aligned}
$$

Thus it follows that $k_i = 0$ for every i.

STEP 2. We verify statement (ii).

According to the equivalence of (1) and (3) from Theorem 1.2, it suffices to show that $\mathbf{H}_{\mathcal{F}}(\mathbb{R}) \neq 0$. Since $\mathrm{r}(H_1^w(\mathcal{F})) < \mathrm{r}(H_1^w(X))$ by assumption, there is a nonzero morphism $\varrho \colon H_1^w(X) \to H_1^w(\mathbb{T})$ which vanishes on $H_1^w(\mathcal{F})$. Choose a continuous base point preserving map $\xi \colon X \to \mathbb{T}$ with $\xi^\sharp = \varrho$. Then $\mathrm{co}(\xi)_* = 0$ by [12, Lemma 2.17, p. 55], so $\mathrm{co}(\xi)$ lifts across the exponential exp to a continuous map $\mathcal{C} \colon \Gamma \times X \to \mathbb{R}$ with $\mathcal{C}(1, z) = 0$, where z is the base point of X. According to [12, Lemma 3.36, p. 84], $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ and we claim that $\mathcal{C} \notin \mathbf{B}_{\mathcal{F}}(\mathbb{R})$. So assume, on the contrary, that $\mathcal{C} = \mathrm{co}(\eta)$ for some continuous base point preserving map $\eta \colon X \to \mathbb{R}$. Then $\xi = \exp \eta$, hence $\varrho = \xi^\sharp = 0$, a contradiction. □

The following corollary of Proposition 5.16 will be our main tool for concrete computations. First let us fix some notation. Let Λ be a Lie group with the identity component Λ_0. Following [13, Section 5], we denote by $(\Lambda/\Lambda_0)^w$ the quotient group of the abelianization of Λ/Λ_0 modulo the torsion subgroup. Thus $(\Lambda/\Lambda_0)^w$ is a torsion-free abelian group.

Corollary 5.17 *Let Γ be a connected Lie group and $\Lambda \subseteq \Gamma$ be a closed cocompact subgroup. Then the following statements hold:*

(i) *if Λ is simply connected then $\mathrm{r}(A_\Gamma) = \mathrm{r}(\pi_1(\Gamma))$,*
(ii) *if $(\Lambda/\Lambda_0)^w \neq 0$ then $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$.*

Proof Consider the homogeneous flow $\mathcal{F} \colon \Gamma \curvearrowright \Gamma/\Lambda$, $\mathcal{F}(\gamma, \delta\Lambda) = \gamma\delta\Lambda$, with $z = \Lambda$ as the base point for Γ/Λ. Then $\mathcal{F}$ is transitive, hence minimal and the transition map $\mathcal{F}_z$ from (2.8) coincides with the canonical projection $p \colon \Gamma \to \Gamma/\Lambda$. If Λ is simply connected then it follows from the long exact sequence of homotopy groups associated to p [35, p. 152] that $p_* \colon \pi_1(\Gamma) \to \pi_1(\Gamma/\Lambda)$ is an isomorphism, hence so is $p^\sharp \colon H_1^w(\Gamma) \to H_1^w(\Gamma/\Lambda)$. Thus $\mathcal{F}$ is topologically free and Proposition 5.16(i) applies.

Further, by [13, Theorem 75],

$$
\mathrm{r}(H_1^w(\Gamma/\Lambda)) - \mathrm{r}(H_1^w(\mathcal{F})) = \mathrm{r}((\Lambda/\Lambda_0)^w).
$$

Consequently, if $(\Lambda/\Lambda_0)^w \neq 0$ then $\mathrm{r}(H_1^w(\mathcal{F})) < \mathrm{r}(H_1^w(\Gamma/\Lambda))$, so $\mathcal{F}$ has a free cycle. By Proposition 5.16(ii), it follows that $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$. □

Now we present some concrete calculations based on Corollary 5.17.

Example 5.18 Consider the special linear group $\Gamma = \mathrm{SL}_2(\mathbb{R})$. Recall (say, from [13, Section 5.7]) that Γ contains a discrete cocompact subgroup Λ with $\Lambda^w \neq 0$. By Corollary 5.17(ii), it follows that $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$. Further, let Δ be the closed subgroup of Γ formed by the matrices

$$\gamma = \begin{bmatrix} t & s \\ 0 & t^{-1} \end{bmatrix}$$

with $s \in \mathbb{R}$ and $t > 0$. It follows from the Iwasawa decomposition of Γ that $\Gamma = \mathrm{SO}(2) \cdot \Delta$, so Γ/Δ is compact. Finally, being homeomorphic to $\mathbb{R}^2$, Δ is simply connected, hence Corollary 5.17(i) yields $\mathrm{r}(A_\Gamma) = \mathrm{r}(\pi_1(\Gamma)) = 1$. In summary, $\pi^1(M(\Gamma)) \cong \mathbb{R} \oplus \mathbb{Z}$ by Theorem 1.2.

Example 5.19 Consider the open subgroup $\Gamma = \mathrm{GL}_2^+(\mathbb{R})$ of the general linear group $\mathrm{GL}_2(\mathbb{R})$, formed by matrices with a positive determinant. Then Γ may be identified with the direct product $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ via the topological isomorphism

$$\Gamma \to \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}, \quad \gamma \mapsto (\det(\gamma)^{-\frac{1}{2}}\gamma, \log\det(\gamma)).$$

It follows, in particular, that $\pi_1(\Gamma) \cong \pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$. Considering now Λ and Δ from Example 5.18 and taking into account the subgroups $\Lambda \times \mathbb{R}$ and $\Delta \times \mathbb{R}$ of Γ, we infer from Corollary 5.17 that $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$ and $\mathrm{r}(A_\Gamma) = \mathrm{r}(\pi_1(\Gamma)) = 1$. Thus $\pi^1(M(\Gamma)) \cong \mathbb{R} \oplus \mathbb{Z}$.

Example 5.20 Recall that every noncompact connected abelian Lie group Γ is topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^m$ for some $n \geq 1$ and $m \geq 0$ [45, Theorem 36, p. 111]. The case $m = 0$ was discussed in Example 5.9, so let $m \geq 1$. The subgroups

$$\Lambda = \mathbb{Z}^n \times \mathbb{T}^m \quad \text{and} \quad \Delta = \mathbb{R}^n \times \{1\}$$

of Γ are closed and cocompact, we have $(\Lambda/\Lambda_0)^w \cong \mathbb{Z}^n$ and Δ is simply connected. Since $\pi_1(\Gamma) \cong \mathbb{Z}^m$, it follows from Corollary 5.17 that $\pi^1(M(\Gamma)) \cong \mathbb{R} \oplus \mathbb{Z}^m$.

Example 5.21 Let Γ be the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Denote by X the phase space of the universal minimal flow of $\mathrm{SL}_2(\mathbb{R})$ and recall from Example 5.18 that $\pi^1(X) \neq 0$. Since $\mathrm{SL}_2(\mathbb{R})$ acts on X in a minimal way, the same is true for Γ. Thus we infer from Corollary 5.8 that $\pi^1(M(\Gamma)) \cong \mathbb{R}$.

5.3 Homeomorphism groups of surfaces

In this section we compute the first cohomotopy groups of universal minimal flows $M(\Gamma)$ associated to identity components $\mathcal{H}_0(X)$ of homeomorphism groups of some compact connected surfaces X. We begin with surfaces without boundary. (Recall that $\chi(X)$ is the Euler characteristic of X. For the properties of X and $\mathcal{H}_0(X)$ used below, see Sects. 2.11 and 2.12.)

Proposition 5.22 *Let X be a compact connected surface without boundary and with $\chi(X) < 0$, and let $\Gamma = \mathcal{H}_0(X)$. Then $\pi^1(M(\Gamma)) \cong \mathbb{R}$.*

Remark 5.23 Condition $\chi(X) < 0$ excludes the sphere, the projective plane, the torus and the Klein bottle, leaving the n-fold connected sum of tori for $n \geq 2$ and the m-fold connected sum of projective planes for $m \geq 3$.

Proof of Proposition 5.22 By our assumptions on X, Γ is contractible, hence simply connected. Since the evaluation flow $\mathcal{F}\colon \Gamma \curvearrowright X$ is transitive and $\pi^1(X) \cong H_1^w(X) \neq 0$, the result follows from Corollary 5.8. $\square$

Now we focus our attention on surfaces with boundary. To put our results in a proper context, we first prove the following corollary of [27, Theorem 1.1].

Proposition 5.24 *Let X be a compact connected manifold with nonempty boundary and of dimension at least* 3. *Then the universal minimal flow of $\Gamma = \mathcal{H}_0(X)$ is not metrizable.*

Proof Choose a boundary component Y of X. Clearly, Y is a compact connected manifold without boundary of dimension at least 2. Consider the topological morphism

$$\varphi\colon \Gamma \to \mathcal{H}(Y), \quad T \mapsto T|_Y,$$

and set $G = \operatorname{im}(\varphi)$. Then G is a locally transitive subgroup of $\mathcal{H}(Y)$ (in the sense as recalled in Sect. 1.2), hence the phase space $M(G)$ of its universal minimal flow is not metrizable according to [27, Theorem 1.1]. Since Γ acts in a minimal way on $M(G)$, the proposition follows. $\square$

Proposition 5.25 *Let S be a compact connected surface with nonempty boundary and $\Gamma = \mathcal{H}_0(S)$. Assume that S is not homeomorphic to the disc, the annulus and the Möbius band. Then $\pi^1(M(\Gamma)) \cong \mathbb{R}$.*

Proof It follows from our assumptions on S that Γ is contractible, hence simply connected. By compactness of S, the boundary components $T_1, \dots, T_m$ ($m \geq 1$) of S are simple closed curves and each of them is an invariant set for the evaluation flow $\Gamma \curvearrowright S$. Since elements of Γ may rotate these components independently, the diagonal action of Γ on S^m restricts to a transitive action on $X = T_1 \times \cdots \times T_m$. Finally, $\pi^1(X) \cong \mathbb{Z}^m$, so Corollary 5.8 applies. $\square$

In the following example $\mathbb{A}$ denotes the closed annulus $\mathbb{T} \times [0, 1]$.

Example 5.26 Recall from Sect. 2.11 that $\Gamma = \mathcal{H}_0(\mathbb{A})$ is homotopy equivalent to $\mathbb{T}$, hence $\pi_1(\Gamma) \cong \mathbb{Z}$. Let T, T' be the boundary components of $\mathbb{A}$. Identify T with $\mathbb{T}$ in the natural way and consider the transitive flow $\mathcal{F}\colon \Gamma \curvearrowright \mathbb{T}$ obtained by restricting the evaluation flow $\Gamma \curvearrowright \mathbb{A}$. Use $z = 1$ as the base point of $\mathbb{T}$. Since the transition map $\mathcal{F}_z\colon \Gamma \to \mathbb{T}$ (see (2.8)) has a continuous global cross section (mapping each $x \in \mathbb{T}$ to the horizontal rotation of $\mathbb{A}$ by x), $(\mathcal{F}_z)_*\colon \pi_1(\Gamma) \to \pi_1(\mathbb{T})$ is an epimorphism, hence an isomorphism. Further, identify $T \times T'$ with $\mathbb{T}^2$ and consider the transitive action $\mathcal{G}\colon \Gamma \curvearrowright \mathbb{T}^2$ as defined in our proof of Proposition 5.25. Use $z = 1$ as the base point of $\mathbb{T}^2$. Then $\mathrm{r}(\operatorname{im}((\mathcal{G}_z)_*)) \leq \mathrm{r}(\pi_1(\Gamma)) < \mathrm{r}(\pi_1(\mathbb{T}^2))$. Now, by applying to $\mathcal{F}$ and $\mathcal{G}$ ideas analogous to those from Proposition 5.16, we obtain $\mathrm{r}(A_\Gamma) = 1$ and $\pi^1(\mathcal{U}(\Gamma)) \cong \mathbb{R}$. Thus $\pi^1(M(\Gamma)) \cong \mathbb{R} \oplus \mathbb{Z}$ according to Theorem 1.2.

5.4 Orbits of homeomorphism groups of surfaces

Our aim in this last section is to prove Theorem 1.3. As was hinted in the Introduction (see Sect. 1.4), the desired minimal flow on $X \times \mathbb{T}$ will be constructed as a group extension $\mathcal{F}_{\mathcal{C}}$ of the evaluation flow $\mathcal{F}$ of $\Gamma = \mathcal{H}_0(X)$ on X (see (2.2) and (2.11)), via a minimal cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$. Since Γ is not locally compact, the compact-open topology on $\mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ does not serve and $\mathcal{C}$ thus can not be searched for by any familiar Baire category argument. Instead, we obtain the required cocycle by first lifting an appropriate coboundary $\operatorname{co}(\xi) \in \mathbf{B}_{\mathcal{F}}(\mathbb{T})$

across the exponential exp (which is also allowed for large topological groups on the account of [12, Theorem 3.38, p. 84]) and then project the cocycle $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ thus obtained via a carefully chosen character $\chi \in \mathbb{R}^*$, relying on our Corollary 3.18.

The following lemma is well known but we include its short proof for the reader's convenience. Recall that a morphism between minimal flows is feebly open in the sense that the image of every nonempty open set has a nonempty interior. (Also recall our notation $\mathcal{O}_{\mathcal{F}}(x)$ for the orbit of x under $\mathcal{F}$.)

Lemma 5.27 *Let $\mathcal{F}, \mathcal{G}$ be minimal Γ-flows with phase spaces X, Y, respectively, $h \colon \mathcal{G} \to \mathcal{F}$ be a morphism, $y \in Y$ and $x = h(y)$. If $\mathcal{O}_{\mathcal{F}}(x)$ is of first category in X then $\mathcal{O}_{\mathcal{G}}(y)$ is of first category in Y.*

Proof By assumption, there exist closed sets $C_n \subseteq X$ $(n \in \mathbb{N})$ with empty interiors such that $\mathcal{O}_{\mathcal{F}}(x) \subseteq \bigcup_{n=1}^{\infty} C_n$. Using that h is feebly open, we infer that each of the closed sets $h^{-1}(C_n) \subseteq Y$ has empty interior in Y. Since

$$\mathcal{O}_{\mathcal{G}}(y) \subseteq h^{-1}(\mathcal{O}_{\mathcal{F}}(x)) \subseteq \bigcup_{n=1}^{\infty} h^{-1}(C_n),$$

the statement of the lemma follows. □

Given a flow $\mathcal{F} \colon \Gamma \curvearrowright X$ and $x \in X$, we denote by Γ_x the *isotropy group* of x,

$$\Gamma_x := \{\alpha \in \Gamma \colon \mathcal{F}(\alpha, x) = x\}. \tag{5.3}$$

Lemma 5.28 *Let X be a compact connected surface without boundary and with $\chi(X) < 0$, let $\Gamma = \mathcal{H}_0(X)$ and $\mathcal{F} \colon \Gamma \curvearrowright X$ be the evaluation flow. Then there exists $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ minimal and such that $\mathcal{C}(\Gamma_x \times \{x\})$ is countable for every $x \in X$.*

Proof We have $\pi^1(X) \cong H_1^w(X) \neq 0$ by assumption, so there is a continuous map $\xi \colon X \to \mathbb{T}$ representing a nonzero element of $\pi^1(X)$. Since Γ is locally contractible and simply connected, $\mathrm{co}(\xi) = \mathrm{Exp}_{\mathcal{F}}(\mathcal{D})$ for some $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ according to [12, Theorem 3.38, p. 84]. By our choice of ξ, it does not lift across exp to any continuous map $X \to \mathbb{R}$, so $\mathcal{D}$ is not a coboundary. In view of Corollary 3.18, it follows that $\mathcal{D}^{\times}$ is countable (see Definition 3.6), hence $\mathcal{C} = \chi\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ is minimal for some $\chi \in \mathbb{R}^*$. Now let $x \in X$. Then by (5.3) we have for every $\alpha \in \Gamma_x$,

$$\mathrm{co}(\xi)(\alpha, x) = \xi(\alpha(x))\xi(x)^{-1} = \xi(x)\xi(x)^{-1} = 1,$$

hence $\mathcal{D}(\alpha, x) \in \mathbb{Z}$. Thus $\mathcal{C}(\Gamma_x \times \{x\})$ is contained in the countable set $\chi(\mathbb{Z})$. □

The following lemma is probably well known. We include its proof for the sake of completeness.

Lemma 5.29 *Let X be a compact connected surface without boundary, $\Gamma = \mathcal{H}_0(X)$ and $x \in X$. Then the projection*

$$\pi_x \colon \Gamma \to X, \quad \gamma \mapsto \gamma(x),$$

has continuous local cross sections.

Proof Let $D \subseteq X$ be a closed topological disc containing x in its interior $\mathrm{Int}(D)$. Since interiors of such discs cover X, it is sufficient to find a continuous local cross section for

π_x defined on $\mathrm{Int}(D)$. To simplify notation, we shall identify D with the unit disc $\mathbb{D}^2 \subseteq \mathbb{R}^2$ centered at the origin 0 in a way that x corresponds to 0. Given $y \in \mathrm{Int}(D)$, consider the map

$$\tau_y : D \to D, \quad v \mapsto v + (1 - \|v\|)y,$$

where $\|v\|$ is the euclidean norm of v. It follows by an elementary verification that τ_y is a homeomorphism fixing the boundary of D. As such, it extends identically outside D to a homeomorphism $\tau_y : X \to X$. Now $\tau_y(v)$ is jointly continuous in $y \in \mathrm{Int}(D)$ and $v \in X$, hence the assignment $y \mapsto \tau_y$ defines a continuous map $\varrho : \mathrm{Int}(D) \to \mathcal{H}(X)$. Moreover, since $\mathrm{Int}(D)$ is path-connected and $\tau_0 = \mathrm{id}_X$, ϱ takes its values in Γ. Finally, for every $y \in \mathrm{Int}(D)$ we have

$$\pi_x \varrho(y) = \pi_x(\tau_y) = \tau_y(0) = y,$$

which means that ϱ is a local cross section for π_x. □

Given a compact space X and $v \in \mathbb{T}$, we denote by R_v the vertical rotation of $X \times \mathbb{T}$ by v. If $\mathcal{F} : \Gamma \curvearrowright X$ is a minimal flow and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ then R_v is an automorphism of $\mathcal{F}_{\mathcal{C}}$.

Proposition 5.30 *Let X be a compact connected surface without boundary and with $\chi(X) < 0$, let $\Gamma = \mathcal{H}_0(X)$ and $\mathcal{F} : \Gamma \curvearrowright X$ be the evaluation flow. Then there exists $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T})$ minimal and such that all orbits of $\mathcal{F}_{\mathcal{C}}$ are of first category in $X \times \mathbb{T}$.*

Proof We show that any $\mathcal{C}$ provided by Lemma 5.28 will do. So fix $x \in X$ and $v \in \mathbb{T}$. Recall notation (5.3) and write $\mathcal{C}(\Gamma_x \times \{x\}) = \{v_n : n \in \mathbb{N}\}$. By Lemma 5.29, we may cover X by closed topological discs $D_1, \ldots, D_m$, each of which supports a continuous local cross section $\varrho_i : D_i \to \Gamma$ of π_x. Being continuous and injective, the maps ϱ_i are homeomorphisms onto their images, so $\varrho_i(D_i)$ are closed topological discs in Γ. Let K be their union. Since $\pi_x(K) = X$, we have $\Gamma = K \cdot \Gamma_x$. Now let $\gamma \in \Gamma$ and write $\gamma = \delta\alpha$, where $\delta \in K$ and $\alpha \in \Gamma_x$. Choose i with $\delta \in \varrho_i(D_i)$ and $n \in \mathbb{N}$ with $\mathcal{C}(\alpha, x) = v_n$. Then

$$\begin{aligned}\mathcal{F}_{\mathcal{C}}(\gamma, (x, v)) &= \mathcal{F}_{\mathcal{C}}(\delta, \mathcal{F}_{\mathcal{C}}(\alpha, (x, v))) = \mathcal{F}_{\mathcal{C}}(\delta, (\mathcal{F}(\alpha, x), \mathcal{C}(\alpha, x)v)) = \mathcal{F}_{\mathcal{C}}(\delta, (x, v_n v)) \\ &= R_{v_n v}\mathcal{F}_{\mathcal{C}}(\delta, (x, 1)) \in R_{v_n v}\mathcal{F}_{\mathcal{C}}(\varrho_i(D_i) \times \{(x, 1)\}).\end{aligned}$$

Thus it follows that

$$\mathcal{O}_{\mathcal{F}_{\mathcal{C}}}(x, v) \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{m} R_{v_n v}\mathcal{F}_{\mathcal{C}}(\varrho_i(D_i) \times \{(x, 1)\}),$$

hence we need only verify that the closed sets $C_i := \mathcal{F}_{\mathcal{C}}(\varrho_i(D_i) \times \{(x, 1)\})$ have empty interiors in $X \times \mathbb{T}$. We show that each C_i is in fact a closed topological disc. So fix i and consider the continuous map

$$h_i : D_i \to X \times \mathbb{T}, \quad y \mapsto \mathcal{F}_{\mathcal{C}}(\varrho_i(y), (x, 1)).$$

If pr denotes the projection $X \times \mathbb{T} \to X$ then for every $y \in D_i$ we have

$$\mathrm{pr}\, h_i(y) = \mathcal{F}(\varrho_i(y), x) = \pi_x \varrho_i(y) = y.$$

Thus it follows that h_i is injective, hence a homeomorphism onto its image, so $h_i(D_i) = C_i$ is truly a closed topological disc. □

Proof of Theorem 1.3 The first assertion of the theorem follows from Proposition 5.30. The second assertion is a consequence of the first assertion, Lemma 5.27 and the universality of $\mathcal{U}(\Gamma)$. □

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