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Minimal extensions in smooth dynamics

Matúš Dirbák[1]

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Abstract

A classical result of Fathi and Herman from 1977 states that a smooth compact connected manifold without boundary admitting a locally free action of a 1-torus, respectively, an almost free action of a 2-torus, admits a minimal diffeomorphism, respectively, a minimal flow. In the first part of our paper we study the existence of locally free and almost free actions of tori on homogeneous spaces of compact connected Lie groups, thus providing new examples of spaces admitting minimal diffeomorphisms or flows. In the second part we combine the ideas of Fathi and Herman with our recent ideas to study the existence of minimal skew products over certain minimal flows with general connected Lie groups as acting groups. Our results apply to so called flows with free cycles. In the last part of our work we study the existence of free cycles in homogeneous flows.

Keywords Minimal flow · Minimal extension · Homogeneous space · Homogeneous action · Free cycle

Mathematics Subject Classification 37B05 · 37C05 · 37C10 · 37C85

1 Introduction

1.1 Some conventions

All maps considered in this paper are implicitly assumed (just) continuous. If we use or claim smoothness or analyticity, we express it explicitly. Likewise, by a *manifold* we

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✉ Matúš Dirbák
Matus.Dirbak@umb.sk

[1] Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

understand a topological manifold, unless explicitly adding smoothness or analyticity assumptions. By the term *analytic* we shall understand *real-analytic* and *smooth* will mean *of type* C^∞. By a *dynamical system* $\mathcal{F}\colon \Gamma \curvearrowright X$ we understand a continuous action of a Lie group Γ on a compact connected manifold X without boundary. We call $\mathcal{F}$ a *flow* when its acting group Γ is connected. A dynamical system is called *minimal* if all its orbits are dense.

An ℓ-*torus* is a Lie group T_ℓ isomorphic to $\mathbb{T}^\ell$. Depending on the context, we shall write the group operations on T_ℓ either multiplicatively or additively. The former notation is usually adopted when T_ℓ is considered as a subgroup of some non-abelian group G, the latter is often used when T_ℓ is considered as a group on its own.

1.2 The approximation by conjugation method of Anosov and Katok

The approximation by conjugation method invented by Anosov and Katok in 1970 is a useful and powerful method of constructing dynamical systems with unexpected properties. In their seminal paper [1], where the method was described for the first time, the authors were interested in ergodic diffeomorphisms on manifolds and proved, in particular, that there is an area-preserving ergodic diffeomorphism of the disk. More generally, their method applied to compact manifolds admitting a nontrivial smooth action of $\mathbb{T}^1$.

Despite the technical difficulties one often encounters when applying the method or its variations, it has nevertheless a simple geometric interpretation. Start with a family of simple dynamical systems $\mathcal{F}_i$ $(i \in I)$ on a given manifold X. After performing careful "squeezing and stretching" operations on X, some of the systems $\mathcal{F}_i$ may begin to resemble a one with more complicated dynamics. When the deformations of X are performed via homeomorphisms $h\colon X \to X$, the dynamics we observe are those of the conjugate systems $h\mathcal{F}_i h^{-1}$. To obtain, at last, a dynamical system $\mathcal{F}$ that truly exhibits complicated dynamics, we use the conjugate systems as its approximations.

The approximation by conjugation method became very soon a classical part of the theory of dynamical systems and has been ever since its discovery a source of many important constructions, both in ergodic theory and topological dynamics [11]. In fact, shortly after discovering the method, Katok realized that it fits well not only with ergodicity but also with minimality, which is a topological property in its nature. In [17] he announced the following result.

Theorem 1 [17] *Let Z be a smooth compact connected manifold without boundary admitting a smooth free action of $\mathbb{T}^1$, respectively, of $\mathbb{T}^2$. Then Z admits a minimal diffeomorphism, respectively, a minimal smooth flow $\mathbb{R} \curvearrowright Z$.*

1.3 Minimal dynamical systems

Minimality is one of the fundamental notions of the topological theory of dynamical systems. Since every dynamical system has a minimal set by Zorn's lemma, understanding minimality is an important step towards understanding general dynamical systems. A fundamental problem of the general theory consists of determining which spaces support minimal dynamics. When constructing a minimal system on a space X, one may try to exploit its additional structure (say, algebraic, geometric, differentiable) and construct a minimal system on X respecting this structure. When X has the form of a direct product, $X = Y \times Z$, then searching among the systems on X having the form of a direct product seems a natural first choice [6]. Since this attempt requires that both Y and Z admit minimal dynamics, it is too restrictive for more general constructions. Thus one may consider skew products as an intermediate notion between direct products and general systems on X. Recall that a skew product on X is a system that factors onto a system on Y via the projection $X \to Y$. Minimality of such a system implies minimality of the corresponding factor on Y but it does not require that Z admit a minimal dynamical system.

Being a property of type G_δ, minimality is well suited for the Baire category argument and, as was mentioned before, it fits well with the approximation by conjugation method. Moreover, it was further realized that the method can serve to construct dynamical systems in the form of skew products with various dynamical properties lifted from the base to the extension, both in the topological category [9] and in the smooth category [25]. The technical difficulties one encounters when lifting minimality depend, on the one hand, on the acting groups of the considered systems, it being usually more difficult to extend minimal systems with more abstract acting groups (say, beyond abelian, amenable or simply connected) [4, 5]. On the other hand, the difficulty is influenced by regularity of the considered systems: the technicalities are sometimes finer in the smooth category than in the topological one, not to mention the category of analytic maps [8].

1.4 Two results of Fathi and Herman

Some time after Theorem 1 has been announced, Fathi and Herman obtained analogous but stronger results under more relaxed assumptions on the toral actions, namely Theorems 3 and 8 below.

Definition 2 A continuous action of a torus T on a compact metrizable space Z is called *locally free* if the isotropy group of each $z \in Z$ is finite.

Theorem 3 [7, Théorème 3] *Let Z be a compact connected smooth manifold without boundary admitting a smooth locally free action of $\mathbb{T}^1$. Then Z admits a strictly ergodic (hence minimal) diffeomorphism.*

Several examples of (classes of) manifolds fitting into the setting of Theorem 3 (and Theorem 8 below) are presented in [7, Exemple 3.9]. We describe other examples in our Sects. 2 and 4. Thus, on one hand, Theorem 3 can serve as a useful source of examples of manifolds admitting minimal diffeomorphisms. On the other hand, the result bears a potential to be useful also when proving more theoretical results, such as Corollary 4 below. Indeed, recall that the periodic point property of a manifold is an obstruction for its minimality. A weak version of the converse implication can be proved in the form of the following reformulation of Theorem 3.

Corollary 4 *Let Z be a compact connected riemannian manifold without boundary. If Z admits no minimal diffeomorphisms then it has the periodic point property for isometries.*

Proof Let G be the group of isometries of Z, equipped with the compact-open topology. By [19, Theorem 1.2, p. 39], G carries the structure of a compact Lie group in such a way that the natural action of G on Z is smooth. Now assume that $g \in G$ has no periodic points. Then the closed subgroup H of G generated by g is infinite and abelian, hence its identity component T is a non-trivial torus. By passing to an appropriate iterate of g if necessary, we may assume that $g \in T$. Let d denote the distance function on Z. Since g has no fixed points, we have $\min_{z \in Z} d(g(z), z) > 0$, hence the same is true for each $h \in T$ sufficiently close to g. Choose such an h which generates topologically a 1-torus T_1 in T. Given $z \in Z$, let I_z be the isotropy group of z with respect to the action of T_1 on Z. Since I_z does not contain h, it is a proper closed subgroup of T_1, hence is finite. Thus T_1 acts smoothly and locally freely on Z, meaning that Z admits a minimal diffeomorphism by Theorem 3. □

The implication from Corollary 4 can not be reversed, an obvious reason being that while the isometry group of a manifold Z depends on the riemannian metric, the minimality of Z does not. One class of manifolds mentioned in [7, Exemple 3.9] consists of the homogeneous spaces of compact connected Lie groups modulo finite subgroups. This generalizes immediately as follows.

Corollary 5 *Let Γ be a connected Lie group with a nontrivial maximal compact subgroup and $\Lambda \subseteq \Gamma$ be a uniform lattice. Then the homogeneous space Γ/Λ admits a minimal diffeomorphism.*

Proof Since maximal compact subgroups of connected Lie groups are connected, Γ contains a 1-torus T_1 by assumptions. The homogeneous action $T_1 \curvearrowright \Gamma/\Lambda, \vartheta\cdot(\gamma\Lambda) = (\vartheta\gamma)\Lambda$, is smooth and its isotropy groups $T_1 \cap \gamma\Lambda\gamma^{-1}$ $(\gamma \in \Gamma)$ are finite (cf. Lemma 9), hence Theorem 3 applies. □

The family of Lie groups Γ covered by Corollary 5 includes, for instance, the special linear groups $\mathrm{SL}(n, \mathbb{R})$ $(n \geq 2)$. It is certainly not obvious, to what extent the mentioned example from [7] generalizes to homogeneous spaces of compact connected Lie groups modulo general closed subgroups. This problem is studied in our Sect. 2.

While the existence of a locally free action of a 1-torus is sufficient to yield a minimal diffeomorphism, a property stronger than local freeness is needed to prove an analogous result for flows.

Definition 6 A continuous locally free action of a 2-torus T_2 on a compact metrizable space Z is called *almost free* if there is a 1-torus $T_1 \subseteq T_2$ such that the restricted action of T_1 on Z is free.

Remark 7 We took the liberty of shortening the terminology introduced in [7], where an almost free action was referred to as *une action localement libre spéciale*. By definitions, each free action is almost free and every almost free action is locally free. Examples 56 and 57 in Sect. 4 illustrate (in the setting of homogeneous actions on $\mathrm{U}(4)/\mathrm{U}(2) = \mathrm{V}_2(\mathbb{C}^4)$, with $\mathrm{U}(n)$ being the unitary group of degree n) that these two implications can not be reversed.

Theorem 8 [7, Théorème 4] *Let Z be a compact connected smooth manifold without boundary. If Z admits a smooth almost free action of $\mathbb{T}^2$ then it admits a strictly ergodic (hence minimal) smooth flow $\mathcal{F}\colon \mathbb{R} \curvearrowright Z$.*

Though the minimality parts of Theorems 3 and 8 may appear only as slight strengthenings of Theorem 1 at first glance, we shall see later in this paper that, on the contrary, the very opposite is true: the local or almost freeness assumption on the toral actions make the two results of Fathi and Herman applicable to a much wider class of manifolds.

1.5 The main aims of this paper

Being motivated by Theorems 3 and 8, in Sect. 2 we study the existence of locally free actions of tori on manifolds, concentrating mostly on homogeneous actions on homogeneous spaces of compact connected Lie groups $Z = G/H$. Our main result in this section is Theorem 16, which gives a necessary and sufficient condition for the existence of an ℓ-torus $T_\ell \subseteq G$ with a locally free homogenous action on Z. We continue by presenting concrete examples of manifolds admitting such actions and discuss several natural problems related to Theorem 16.

In Sect. 3 we begin our study of the problem of lifting minimality to extensions in the form of skew products. In Theorem 49 we show that every smooth (respectively, analytic) minimal flow with a free cycle admits a smooth (respectively, analytic) minimal group extension with the fibre being the ℓ-torus $\mathbb{T}^\ell$ ($\ell \in \mathbb{N}$). By a group extension is understood a skew product acting by rotations in the fibres. The notion of free cycles for a minimal flow was introduced in [5] and is recalled in our Sect. 3.4.

In Sect. 4 we continue our study of minimal extensions by considering minimal skew products with a more general fibre Z. In Theorem 58 we show that every smooth minimal flow with a free cycle admits a smooth minimal skew product with the fibre Z being an arbitrary smooth compact connected manifold without boundary admitting a smooth almost free action of $\mathbb{T}^2$. Our proof of this result relies on a combination of Theorem 49 with a variation of the method used by Fathi and Herman to prove their Theorem 8. Theorems 8 and 58 motivate our interest in manifolds Z admitting an almost free action of $\mathbb{T}^2$. Almost free actions are discussed in Sect. 4.1 and examples of manifolds admitting such actions are presented in Sect. 4.4. Similarly to Sect. 2, we focus on homogeneous actions on homogeneous spaces of compact connected Lie groups $Z = G/H$.

We wish to mention that in the statements of our Theorems 49 and 58 there are no explicit restrictions on the acting groups of the extended minimal flows. Thus our methods apply to actions of general connected Lie groups. On the other hand, the existence of free cycles is an important technical assumption in both results.

The last part of our paper, Sect. 5, is devoted to the problem of existence of free cycles in homogeneous actions of connected Lie groups. Since homogeneous flows are smooth (in fact, analytic), they can serve as examples of flows to which our Theorems 49 and 58 apply. We begin by finding a necessary and sufficient condition for the existence of free cycles for transitive homogeneous flows in Theorem 75 and continue by applying it in concrete situations. We also study how the existence of free cycles gets affected by changing the acting group or the phase space of a minimal flow, thus obtaining a sufficient condition for the existence of free cycles in general homogeneous flows.

2 Locally free homogeneous actions of tori

2.1 Homogeneous spaces and actions

In this paper much attention is paid to homogeneous spaces and homogeneous actions of Lie groups. We shall therefore dedicate a few lines to summarizing some basic facts concerning these two notions which will be used in the sequel without further notice. Our main reference here is [20, Chapter 9].

Given a Lie group G and a closed subgroup H of G, consider the set G/H consisting of the left cosets of G modulo H. With the quotient topology, G/H is a topological manifold of dimension $\dim(G) - \dim(H)$. It carries a unique smooth structure such that the natural projection $G \to G/H$ is a smooth submersion. The homogeneous action of G on G/H is defined by $g \cdot (qH) = (gq)H$. It is smooth and transitive with a stable isotropy conjugate to H. Moreover, if T is a closed subgroup of G then T is an embedded submanifold of G, hence the restricted action of T on G/H is also smooth. Finally, if G acts on a smooth manifold Z smoothly and transitively with a stable isotropy conjugate to H then Z is equivariantly diffeomorphic to the homogeneous space G/H.

2.2 Auxiliary lemmas

The proof of the following lemma is elementary and straightforward, hence omitted.

Lemma 9 *Let G be a compact Lie group and H, T be closed subgroups of G. Consider the homogeneous action $T \curvearrowright G/H$. Given $g \in G$, the isotropy group of $gH \in G/H$ is*

$$I_g = T \cap gHg^{-1}.$$

In particular, the following conditions are equivalent:

(i) the action is locally free (respectively, free),
(ii) $T \cap gHg^{-1}$ is finite (respectively, trivial) for every $g \in G$,
(iii) $gTg^{-1} \cap H$ is finite (respectively, trivial) for every $g \in G$.

Remark 10 As an immediate corollary of Lemma 9 let us mention that the (local) freeness of a homogeneous action $T \curvearrowright G/H$ is an invariant of conjugation with respect to both T and H. More precisely, if T acts (locally) freely on G/H and $g \in G$ then gTg^{-1} also acts (locally) freely on G/H and so does T on G/gHg^{-1}. Thus when searching for a (locally) free action or when showing that no such action exists, we may (and often will) switch to conjugates within G to make our arguments more transparent.

Lemma 11 *Let G be a compact Lie group, $H \subseteq G$ be a closed subgroup of G with the identity component H_0 and $T \subseteq G$ be a torus. Then $T \cap H$ is finite if, and only if, so is $T \cap H_0$. Consequently, the homogeneous action $T \curvearrowright G/H$ is locally free if, and only if, so is $T \curvearrowright G/H_0$.*

Proof Denote by ℓ the dimension of T and let d be the order of the finite group H/H_0, so that $h^d \in H_0$ for every $h \in H$. Denote by κ_d the d-endomorphism of T, $\kappa_d(t) = t^d$. Then $\kappa_d(T \cap H) \subseteq T \cap H_0$. Thus

$$T \cap H_0 \subseteq T \cap H \subseteq \kappa_d^{-1}(T \cap H_0)$$

and since κ_d is d^ℓ-to-1, the first statement of the lemma follows.

To verify the second statement we apply Lemma 9(iii). Given $g \in G$, the first part of the lemma applied to the torus gTg^{-1} shows that the finiteness of $gTg^{-1} \cap H$ is equivalent to that of $gTg^{-1} \cap H_0$. □

Lemma 9 explains the difficulty one faces when searching for a torus $T \subseteq G$ with a locally free action on G/H, for it shows that the problem requires ensuring that the intersections of T with the elements of the possibly uncountable family gHg^{-1} ($g \in G$) are finite (Example 12). Lemma 14 below is an important step towards overcoming this obstacle in the sense that it will help us reformulate the problem in terms of only finitely many conditions.

Example 12 Let $1 \leq k < n$ be integers and $G = \mathrm{U}(n)$. We shall view $\mathrm{U}(n-k)$ as a closed subgroup H of $\mathrm{U}(n)$ in the usual way, by identifying each $t \in \mathrm{U}(n-k)$ with the block diagonal matrix $\operatorname{diag}(t, 1)$. We shall also view each element g of $\mathrm{U}(n)$ as a unitary operator on $\mathbb{C}^n$ and consider the linear subspace $\operatorname{Fix}(g)$ of the fixed points of g. The standard basis of $\mathbb{C}^n$ will be denoted by $e_1, \dots, e_n$.

Fix $g \in G$. We show that the equality $gHg^{-1} = H$ occurs precisely when g is a block diagonal matrix of the form $g = \operatorname{diag}(p, q)$ with $p \in \mathrm{U}(n-k)$ and $q \in \mathrm{U}(k)$, that is, if, and only if, $g \in \mathrm{U}(n-k) \times \mathrm{U}(k)$. The "if" part is checked easily. To verify the "only if" part, notice first that $S = \bigcap_{h \in H} \operatorname{Fix}(h)$ is the linear subspace of $\mathbb{C}^n$ spanned by the vectors $e_{n-k+1}, \dots, e_n$. Now the equality $gHg^{-1} = H$ implies that $ghg^{-1} \in H$ for every $h \in H$, whence it follows that $hg^{-1}(e_i) = g^{-1}(e_i)$ for all $h \in H$ and $i > n-k$. Thus $g^{-1}(e_i) \in S$ for $i > n-k$, hence $g^{-1}(S) \subseteq S$ and so $g \in \mathrm{U}(n-k) \times \mathrm{U}(k)$ indeed.

It follows from the previous paragraph that for $g, q \in G$, the equality $gHg^{-1} = qHq^{-1}$ is equivalent to $q \in g(\mathrm{U}(n-k) \times \mathrm{U}(k))$. Consequently, the conjugates gHg^{-1} ($g \in G$) are in a one-to-one correspondence with the elements of the Grassmann manifold $\mathrm{Gr}_k(\mathbb{C}^n) = \mathrm{U}(n)/\left(\mathrm{U}(n-k) \times \mathrm{U}(k)\right)$ (which is uncountable).

Remark 13 In the proof of the following lemma, as well as in other parts of this paper, we shall view elements of $\mathrm{U}(n)$ also as unitary operators on $\mathbb{C}^n$. We mention that for a diagonal matrix $s = \operatorname{diag}(z_1, \dots, z_n) \in \mathrm{U}(n)$ with $z_i \in \mathbb{T}^1$, the eigenvalues of s are precisely its diagonal entries and their geometric multiplicities coincide with the numbers of their occurences on the diagonal.

Lemma 14 *Let G be a compact connected Lie group and $S \subseteq T \subseteq G$ be tori. Then there is a finite family $\mathcal{R}$ of closed subgroups of T with* $\dim(R) \leq \dim(S)$ *for every $R \in \mathcal{R}$ and such that*

$$gSg^{-1} \cap T \subseteq \bigcup \mathcal{R} \tag{2.1}$$

for every $g \in G$.

Proof We proceed in four steps.

Step 1. We impose and justify some simplifications.

First notice that the statement of the lemma, when considered as a property of G, is inherited by closed connected subgroups of G containing T. Since every compact Lie group can be viewed as a topological subgroup of $\mathrm{U}(n)$ for some $n \in \mathbb{N}$ [15, Corollary 2.40, p. 49], we may assume, without loss of generality, that $G = \mathrm{U}(n)$. Similarly, if $\mathcal{R}$ is a collection fulfilling the requirements with regard to T and if $S \subseteq T' \subseteq T$ is a torus then $\mathcal{R}' = \{R \cap T' \colon R \in \mathcal{R}\}$ fulfills the requirements with regard to T'. Thus we may work under the assumption that T is a maximal torus of $\mathrm{U}(n)$. Finally, since all maximal tori in $\mathrm{U}(n)$ are conjugate and conjugation in $\mathrm{U}(n)$ does not affect the cardinality of $\mathcal{R}$ and the dimensions of closed subgroups of $\mathrm{U}(n)$, it is sufficient to consider the standard maximal torus in $\mathrm{U}(n)$,

$$T = \{\mathrm{diag}(z_1, \dots, z_n) \colon z_1, \dots, z_n \in \mathbb{T}^1\}.$$

Step 2. The group S_n of the permutations of the set $\{1, \dots, n\}$ can be identified with a subgroup of $\mathrm{U}(n)$ by letting each $\sigma \in S_n$ act on the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ by permutation of indices; in symbols, $\sigma(e_i) = e_{\sigma(i)}$ for every i. Given $g \in \mathrm{U}(n)$ and $s \in S$ with $gsg^{-1} \in T$, we show that $gsg^{-1} = \sigma^{-1}s\sigma$ for some $\sigma \in S_n$.

Write $s = \mathrm{diag}(z_1, \dots, z_n)$ and $gsg^{-1} = \mathrm{diag}(v_1, \dots, v_n)$. Since the eigenvalues of a unitary operator as well as their geometric multiplicities are conjugacy invariants, it follows that $(v_1, \dots, v_n)$ is a permutation of $(z_1, \dots, z_n)$. That is, we have $\sigma \in S_n$ with $v_i = z_{\sigma(i)}$ for every i. We show that $gsg^{-1} = \sigma^{-1}s\sigma$. Indeed, for every $i \in \{1, \dots, n\}$,

$$\sigma^{-1}s\sigma(e_i) = \sigma^{-1}s(e_{\sigma(i)}) = \sigma^{-1}(z_{\sigma(i)}e_{\sigma(i)}) = z_{\sigma(i)}e_i = v_ie_i = gsg^{-1}(e_i). \tag{2.2}$$

Step 3. We mention that S_n is the Weyl group of T in $\mathrm{U}(n)$, hence $\sigma^{-1}T\sigma = T$ for every $\sigma \in S_n$. We check this equality for the sake of completeness.

Clearly, it suffices to check the inclusion "$\subseteq$". So fix $\sigma \in S_n$, choose $t \in T$ and write $t = \mathrm{diag}(z_1, \dots, z_n)$. Then $t(e_i) = z_ie_i$, hence, similarly to (2.2), $\sigma^{-1}t\sigma(e_i) = z_{\sigma(i)}e_i$ for every i. Consequently, $\sigma^{-1}t\sigma = \mathrm{diag}(z_{\sigma(1)}, \dots, z_{\sigma(n)}) \in T$.

Step 4. We prove the lemma by showing that the family $\mathcal{R}$ consisting of the conjugates $\sigma^{-1}S\sigma$ with $\sigma \in S_n$, fulfills the required conditions.

By Step 3, all elements of $\mathcal{R}$ are closed subgroups of T with dimensions equal to that of S and the family $\mathcal{R}$ is finite since S_n is finite. To verify (2.1) fix $g \in \mathrm{U}(n)$ and $s \in S$ with $gsg^{-1} \in T$. In view of Step 2 there is $\sigma \in S_n$ such that $gsg^{-1} = \sigma^{-1}s\sigma$. Then

$$gsg^{-1} = \sigma^{-1}s\sigma \in \sigma^{-1}S\sigma \subseteq \bigcup \mathcal{R},$$

as was to be shown. □

Remark 15 It follows from our proof of Lemma 14 that the set $\mathcal{R}$ can always be chosen in such a way that its cardinality does not exceed $n!$, where n is the smallest positive integer such that G embeds into $\mathrm{U}(n)$. Notice that this upper bound depends only on G, not on S or T.

2.3 A general result

The following theorem is our main result in this section. Before formulating it recall that the rank of a compact Lie group G, denoted by $\operatorname{rank}(G)$, is the dimension of its maximal torus.

Theorem 16 *Let G be a compact connected Lie group and H be a closed subgroup of G. Given $\ell \in \mathbb{N}$, the following conditions are equivalent:*

(1) *there is an ℓ-torus $T_\ell \subseteq G$ whose homogeneous action $T_\ell \curvearrowright G/H$ is locally free,*
(2) $\operatorname{rank}(G) - \operatorname{rank}(H) \geq \ell$.

Remark 17 Notice that condition (2) relies only on intrinsic properties of G and H. The way in which H sits within G plays no role.

In our proof of Theorem 16 we shall deal with the topological models of Lie algebras. Given a torus T, we regard elements of its Lie algebra $\mathfrak{t}$ as the 1-parameter subgroups $\varphi \colon \mathbb{R} \to T$. The exponential $\exp \colon \mathfrak{t} \to T$ is then defined by $\exp(\varphi) = \varphi(1)$. If $T = \mathbb{T}^n$ with $n \in \mathbb{N}$ then $\mathfrak{t}$ can be naturally identified with $\mathbb{R}^n$ by assigning to each $(s_1, \ldots, s_n) \in \mathbb{R}^n$ the 1-parameter subgroup

$$\varphi(t) = \left(e^{i2\pi s_1 t}, \ldots, e^{i2\pi s_n t}\right).$$

Notice that if $R \subseteq T$ is a torus, $\mathfrak{r} \subseteq \mathfrak{t}$ is a linear subspace and $\exp(\mathfrak{r}) = R$, then $\mathfrak{r}$ is the Lie algebra of R.

Proof of Theorem 16 Recall that H and its identity component H_0 have the same rank. Therefore, in view of Lemma 11, it is sufficient to consider the case of H connected.

We verify implication (2) $\Rightarrow$ (1). We begin by fixing some necessary notation and then continue in six steps.

Let S be a maximal torus of H and $T \supseteq S$ be a maximal torus of G. Then

$$\dim(T) - \dim(S) \geq \ell$$

by assumption. Write $\mathfrak{t}$ for the Lie algebra of T. We shall consider the finite family $\mathcal{R}$ from Lemma 14. For each $R \in \mathcal{R}$ we denote by R_0 the identity component of R. Clearly, R_0 is a (possibly trivial) torus in T with $\dim(R_0) = \dim(R) \leq \dim(S)$.

STEP A. Given tori $P, P' \subseteq T$ with Lie algebras $\mathfrak{p}, \mathfrak{p}' \subseteq \mathfrak{t}$, we recall that $P \cap P'$ is finite if, and only if, $\mathfrak{p} \cap \mathfrak{p}' = 0$.

Indeed, the intersection $\mathfrak{p} \cap \mathfrak{p}'$ is the Lie algebra of $P \cap P'$ and, as such, it vanishes if, and only if, $P \cap P'$ has trivial identity component. This amounts to finiteness of $P \cap P'$ by compactness.

STEP B. Write $n = \operatorname{rank}(G) = \dim(T)$. Given $0 \leq k \leq n$, denote by $\mathcal{T}_k$ the set of all Lie subalgebras of $\mathfrak{t}$ which are Lie algebras of the k-tori in T. We show that for $k < n$ and for every finite set $\mathcal{F}_k \subseteq \bigcup_{i=0}^{k} \mathcal{T}_i$ there is $\mathfrak{t}_1 \in \mathcal{T}_1$ such that $\mathfrak{f} \cap \mathfrak{t}_1 = 0$ for each $\mathfrak{f} \in \mathcal{F}_k$.

Under the usual identifications $T = \mathbb{T}^n$ and $\mathfrak{t} = \mathbb{R}^n$, every non-zero vector from $\mathbb{Q}^n$ generates an element of $\mathcal{T}_1$. Consequently, vectors generating algebras $\mathfrak{t}_1 \in \mathcal{T}_1$ are dense in $\mathfrak{t}$ under the norm topology. Now, by a dimension argument, each element of $\mathcal{F}_k$ is nowhere dense in $\mathfrak{t}$. It follows that $\mathfrak{t} \setminus \bigcup \mathcal{F}_k$ is nonempty open, hence, as noticed above, it contains a vector generating an element $\mathfrak{t}_1 \in \mathcal{T}_1$. Clearly, $\mathfrak{f} \cap \mathfrak{t}_1 = 0$ for every $\mathfrak{f} \in \mathcal{F}_k$.

STEP C. Fix $j, k \geq 0$ with $j + k \leq n$ and $\mathfrak{t}_j \in \mathcal{T}_j$, $\mathfrak{t}_k \in \mathcal{T}_k$ with $\mathfrak{t}_j \cap \mathfrak{t}_k = 0$. We recall that $\mathfrak{t}_j \oplus \mathfrak{t}_k \in \mathcal{T}_{j+k}$.

Let $\exp\colon \mathfrak{t} \to T$ be the exponential and let $T_j, T_k \subseteq T$ be the tori corresponding to $\mathfrak{t}_j, \mathfrak{t}_k$, respectively. Clearly, $T_j T_k$ is also a torus in T. Since $\exp(\mathfrak{t}_j \oplus \mathfrak{t}_k) = T_j T_k$, we infer that $\mathfrak{t}_j \oplus \mathfrak{t}_k$ is the Lie algebra of $T_j T_k$. Finally,

$$\dim(T_j T_k) = \dim(\mathfrak{t}_j \oplus \mathfrak{t}_k) = j + k,$$

hence $\mathfrak{t}_j \oplus \mathfrak{t}_k \in \mathcal{T}_{j+k}$.

STEP D. Given $R \in \mathcal{R}$, write $\mathfrak{r}_R \subseteq \mathfrak{t}$ for the Lie algebra of R_0. We find $\mathfrak{t}_\ell \in \mathcal{T}_\ell$ with $\mathfrak{t}_\ell \cap \mathfrak{r}_R = 0$ for every $R \in \mathcal{R}$.

Let d be the dimension of S. Since the set $\{\mathfrak{r}_R \colon R \in \mathcal{R}\} \subseteq \bigcup_{i=0}^{d} \mathcal{T}_i$ is finite and $d + \ell \leq n$ by assumption, we may use successive applications of Steps B and C to find $\mathfrak{t}_1^{(1)}, \ldots, \mathfrak{t}_1^{(\ell)} \in \mathcal{T}_1$ with

$$\left(\mathfrak{r}_R \oplus \mathfrak{t}_1^{(1)} \oplus \mathfrak{t}_1^{(2)} \oplus \cdots \oplus \mathfrak{t}_1^{(i-1)}\right) \cap \mathfrak{t}_1^{(i)} = 0$$

for all $1 \leq i \leq \ell$. In view of Step C, we may conclude by letting

$$\mathfrak{t}_\ell = \mathfrak{t}_1^{(1)} \oplus \mathfrak{t}_1^{(2)} \oplus \cdots \oplus \mathfrak{t}_1^{(\ell)}.$$

STEP E. Let T_ℓ be the ℓ-torus in T with the Lie algebra $\mathfrak{t}_\ell$. We show that

$$T_\ell \cap gHg^{-1} \subseteq \bigcup_{R \in \mathcal{R}} T_\ell \cap R$$

for every $g \in G$.

Fix $g \in G$. Since S is a maximal torus of H, we have $H = \bigcup_{h \in H} hSh^{-1}$, hence

$$gHg^{-1} = \bigcup_{h \in H} ghS(gh)^{-1}.$$

Consequently, by our choice of $\mathcal{R}$,

$$T_\ell \cap gHg^{-1} = \bigcup_{h\in H} \left(T_\ell \cap ghS(gh)^{-1}\right) \subseteq \bigcup_{R\in\mathcal{R}} T_\ell \cap R.$$

Step F. We show that the homogeneous action of T_ℓ on G/H is locally free.

In view of Lemma 9 and Step E, it suffices to verify that $T_\ell \cap R$ is finite for every $R \in \mathcal{R}$. By Lemma 11 this is equivalent to finiteness of $T_\ell \cap R_0$, which follows from Steps A and D.

We verify implication (1) $\Rightarrow$ (2). Fix an ℓ-torus $T_\ell \subseteq G$, a maximal torus $T \supseteq T_\ell$ of G and a maximal torus S of H. Since S is contained in a maximal torus of G, $S \subseteq gTg^{-1}$ for some $g \in G$, hence $S' = g^{-1}Sg \subseteq T$. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_\ell, \mathfrak{s}' \subseteq \mathfrak{t}$ be the Lie algebras of T_ℓ, S', respectively.

Now assume that the homogeneous action of T_ℓ on G/H is locally free, that is, let $T_\ell \cap qHq^{-1}$ be finite for every $q \in G$. Then

$$T_\ell \cap S' = T_\ell \cap g^{-1}Sg \subseteq T_\ell \cap g^{-1}Hg$$

is a finite group, hence $\mathfrak{t}_\ell \cap \mathfrak{s}' = 0$ by the argument of Step A above. Similarly to Step C, one shows that $T_\ell S'$ is a torus in T and $\mathfrak{t}_\ell \oplus \mathfrak{s}'$ is its Lie algebra. Consequently,

$$\begin{aligned}\dim(T) &\geq \dim(T_\ell S') = \dim(\mathfrak{t}_\ell \oplus \mathfrak{s}') = \dim(\mathfrak{t}_\ell) + \dim(\mathfrak{s}') = \dim(T_\ell) + \dim(S') \\ &= \dim(T_\ell) + \dim(S).\end{aligned}$$

Thus

$$\operatorname{rank}(G) - \operatorname{rank}(H) = \dim(T) - \dim(S) \geq \dim(T_\ell) = \ell,$$

as was to be shown. □

In connection with Theorem 3, the following special case of Theorem 16 is of particular interest.

Corollary 18 *Let G be a compact connected Lie group and H be a closed subgroup of G. Then the following conditions are equivalent:*

(i) there is a 1-torus $T_1 \subseteq G$ whose homogeneous action $T_1 \curvearrowright G/H$ is locally free,
(ii) $\operatorname{rank}(H) < \operatorname{rank}(G)$,
(iii) H does not contain any maximal torus of G.

Proof The equivalence of (i) and (ii) follows immediately from Theorem 16. Implication (ii)⇒(iii) is clear. To verify the converse, assume that $\operatorname{rank}(H) = \operatorname{rank}(G) = n$ and fix a maximal torus T of H. Being an n-dimensional torus in G, T is a maximal torus also for G. Since $T \subseteq H$, we are done. □

Example 19 Given a compact connected Lie group G, a generalized flag manifold is the homogeneous space of G modulo the centralizer H of a torus $S \subseteq G$ [2, p. 97].

Since H contains every maximal torus of G containing S, in view of (i) and (iii) from Corollary 18 the space G/H does not admit any locally free homogeneous action of a 1-torus in G.

2.4 Examples related to Theorem 16

The purpose of this subsection is to apply Theorem 16 to some important concrete manifolds. In view of Theorem 3, these will also serve as examples of manifolds admitting minimal diffeomorphisms. For basic information on maximal tori and ranks of classical compact matrix Lie groups used below, we refer to [26, Theorem 9.8, p. 142]. Definitions and some elementary properties of the spaces considered in this subsection are discussed in [2, pp. 68–70].

Example 20 (Complex Stiefel manifolds) Let $1 \leq \ell < n$ be integers. By regarding $\mathrm{U}(n-\ell)$ as a subgroup of $\mathrm{U}(n)$ in the usual way, we may consider the corresponding homogeneous space

$$\mathrm{V}_\ell(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}(n-\ell),$$

known as the Stiefel manifold of orthonormal ℓ-frames in $\mathbb{C}^n$. Since $\operatorname{rank}(\mathrm{U}(k)) = k$ for every $k \in \mathbb{N}$, we have

$$\operatorname{rank}(\mathrm{U}(n)) - \operatorname{rank}(\mathrm{U}(n-\ell)) = \ell,$$

hence $V_\ell(\mathbb{C}^n)$ admits a locally free homogeneous action of an ℓ-torus $T_\ell \subseteq \mathrm{U}(n)$ by Theorem 16.

Example 21 (Complex Stiefel manifolds again) Following our discussion from Example 20, consider the homogeneous action of $\mathrm{SU}(n) \subseteq \mathrm{U}(n)$ on $\mathrm{V}_\ell(\mathbb{C}^n)$. The action is transitive with isotropy conjugate to $\mathrm{SU}(n-\ell)$, hence we may identify $\mathrm{V}_\ell(\mathbb{C}^n)$ with the homogeneous space $\mathrm{SU}(n)/\,\mathrm{SU}(n-\ell)$. Since $\operatorname{rank}(\mathrm{SU}(k)) = k-1$ for every $k \in \mathbb{N}$, we have

$$\operatorname{rank}(\mathrm{SU}(n)) - \operatorname{rank}(\mathrm{SU}(n-\ell)) = \ell,$$

so Theorem 16 yields a locally free homogeneous action of an ℓ-torus $T_\ell \subseteq \mathrm{SU}(n)$ on $V_\ell(\mathbb{C}^n)$.

Example 22 (Real Stiefel manifolds) Given integers $1 \leq \ell < n$, consider the Stiefel manifold of orthonormal ℓ-frames in $\mathbb{R}^n$,

$$\mathrm{V}_\ell(\mathbb{R}^n) = \mathrm{SO}(n)/\,\mathrm{SO}(n-\ell).$$

We have $\operatorname{rank}(\mathrm{SO}(k)) = \lfloor k/2 \rfloor$ for every $k \in \mathbb{N}$, hence

$$d = \operatorname{rank}(\mathrm{SO}(n)) - \operatorname{rank}(\mathrm{SO}(n-\ell)) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-\ell}{2} \right\rfloor.$$

Now d equals either $\lfloor \ell/2 \rfloor$ or $\lfloor \ell/2 \rfloor + 1$, the second case occurring if, and only if, n is even and ℓ is odd. Consequently, $d = 0$ only when n is odd and $\ell = 1$. In all the other cases we have $d > 0$ and $\mathrm{V}_\ell(\mathbb{R}^n)$ thus admits a locally free homogeneous action of a d-torus $T_d \subseteq \mathrm{SO}(n)$ by Theorem 16.

Example 23 (Spheres) Given $n \in \mathbb{N}$, the group $\mathrm{SO}(n+1)$ acts smoothly on the n-sphere $\mathbb{S}^n$ via rotations. The action is transitive with isotropy conjugate to $\mathrm{SO}(n)$, hence $\mathbb{S}^n$ can be identified with $\mathrm{SO}(n+1)/\mathrm{SO}(n) = \mathrm{V}_1(\mathbb{R}^{n+1})$. If n is odd then Example 22 yields a locally free homogeneous action of a 1-torus $T_1 \subseteq \mathrm{SO}(n+1)$ on $\mathbb{S}^n$. (Recall that in this case there is in fact a free smooth action of $\mathbb{T}^1$ on $\mathbb{S}^n$, namely a higher-dimensional analogue of the Hopf fibration of $\mathbb{S}^3$, cf. [7, Exemple 3.9.c)].) If n is even then $\mathbb{S}^n$ does not support any smooth locally free action of $\mathbb{T}^1$. Indeed, the even-dimensional spheres admit no minimal diffeomorphisms due to their having the periodic point property [22, Theorem 21.4, p. 119].

Example 24 The argument of Example 20 applies also to the quaternionic Stiefel manifolds

$$\mathrm{V}_\ell(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-\ell).$$

Since $\mathrm{rank}(\mathrm{Sp}(k)) = k$ for every $k \in \mathbb{N}$, $\mathrm{V}_\ell(\mathbb{H}^n)$ admits a locally free homogeneous action of an ℓ-torus $T_\ell \subseteq \mathrm{Sp}(n)$.

Example 25 (Real Grassmann manifolds) Let $1 \leq \ell < n$ be integers. The Grassmann manifold $\mathrm{Gr}_\ell(\mathbb{R}^n)$ of ℓ-planes in $\mathbb{R}^n$ may be identified with the homogeneous space

$$\mathrm{Gr}_\ell(\mathbb{R}^n) = \mathrm{SO}(n)/\,\mathrm{S}(\mathrm{O}(\ell) \times \mathrm{O}(n-\ell)),$$

where

$$\mathrm{S}(\mathrm{O}(\ell) \times \mathrm{O}(n-\ell)) = \big[\mathrm{O}(\ell) \times \mathrm{O}(n-\ell)\big] \cap \mathrm{SO}(n).$$

As recalled in Example 22, $\mathrm{rank}(\mathrm{SO}(k)) = \lfloor k/2 \rfloor$ for every $k \in \mathbb{N}$. Moreover, since the identity component of $\mathrm{S}(\mathrm{O}(\ell) \times \mathrm{O}(n-\ell))$ is $\mathrm{SO}(\ell) \times \mathrm{SO}(n-\ell)$, we have

$$\mathrm{rank}\,(\mathrm{S}(\mathrm{O}(\ell) \times \mathrm{O}(n-\ell))) = \mathrm{rank}(\mathrm{SO}(\ell)) + \mathrm{rank}(\mathrm{SO}(n-\ell)) = \left\lfloor \frac{\ell}{2} \right\rfloor + \left\lfloor \frac{n-\ell}{2} \right\rfloor.$$

Hence

$$d = \mathrm{rank}(\mathrm{SO}(n)) - \mathrm{rank}\,(\mathrm{S}(\mathrm{O}(\ell) \times \mathrm{O}(n-\ell))) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\ell}{2} \right\rfloor - \left\lfloor \frac{n-\ell}{2} \right\rfloor.$$

Now d takes values only 0 and 1, the latter case occuring if, and only if, n is even and ℓ is odd. In such a case $\mathrm{Gr}_\ell(\mathbb{R}^n)$ admits a locally free homogeneous action of a 1-torus $T_1 \subseteq \mathrm{SO}(n)$ by Theorem 16.

Example 26 (Real projective spaces) Given $n \in \mathbb{N}$, the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ is the Grassmann manifold $\mathrm{Gr}_1(\mathbb{R}^{n+1})$. As follows from Example 25, if n is odd then $\mathbb{R}\mathrm{P}^n$ admits a smooth locally free action of $\mathbb{T}^1$. Similarly to the situation in Example 23, the even-dimensional projective spaces $\mathbb{R}\mathrm{P}^n$ do not admit any minimal diffeomorphisms, hence neither they admit any smooth locally free action of $\mathbb{T}^1$. Indeed, having the spheres $\mathbb{S}^n$ as their universal covers, the spaces $\mathbb{R}\mathrm{P}^n$ have the periodic point property for every even n.

Remark 27 In analogy to Example 25 we may consider the complex Grassmann manifolds

$$\mathrm{Gr}_\ell(\mathbb{C}^n) = \mathrm{SU}(n)/\,\mathrm{S}(\mathrm{U}(\ell) \times \mathrm{U}(n-\ell)),$$

where

$$\mathrm{S}(\mathrm{U}(\ell) \times \mathrm{U}(n-\ell)) = \left[\mathrm{U}(\ell) \times \mathrm{U}(n-\ell)\right] \cap \mathrm{SU}(n).$$

Since both $\mathrm{SU}(n)$ and $\mathrm{S}(\mathrm{U}(\ell) \times \mathrm{U}(n-\ell))$ have the same rank $n-1$ (the standard maximal torus of the former group being contained in the latter), by Theorem 16 the spaces $\mathrm{Gr}_\ell(\mathbb{C}^n)$ admit no locally free homogeneous actions of tori in $\mathrm{SU}(n)$.

Example 28 (Real partial flag manifolds) Let $n_1, \ldots, n_k$ be positive integers, $n = \sum_{i=1}^k n_i$ and consider the real partial flag manifold

$$\mathrm{F}_{n_1,\ldots,n_k}(\mathbb{R}) = \mathrm{SO}(n)/\,\mathrm{S}(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where

$$\mathrm{S}(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)) = \mathrm{SO}(n) \cap (\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)).$$

The difference

$$\mathrm{rank}(\mathrm{SO}(n)) - \mathrm{rank}(\mathrm{S}(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k))) = \left\lfloor \frac{n}{2} \right\rfloor - \sum_{i=1}^{k} \left\lfloor \frac{n_i}{2} \right\rfloor$$

is positive if, and only if, at least two of n_i's are odd. In such a case Theorem 16 can be applied to show the existence of a 1-torus in $\mathrm{SO}(n)$ with a locally free homogeneous action on $\mathrm{F}_{n_1,\ldots,n_k}(\mathbb{R})$.

Remark 29 Following Example 28, we may consider the complex partial flag manifold

$$\mathrm{F}_{n_1,\ldots,n_k}(\mathbb{C}) = \mathrm{SU}(n)/\,\mathrm{S}(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)).$$

Since both $\mathrm{SU}(n)$ and $\mathrm{S}(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k))$ have the same rank $n-1$ (the reason being the same as in Remark 27), by Theorem 16 there are no locally free homogeneous actions of tori in $\mathrm{SU}(n)$ on $\mathrm{F}_{n_1,\ldots,n_k}(\mathbb{C})$.

Table 1 Selected homogeneous spaces of exceptional Lie groups

G	E_6	E_6	E_7	E_7	E_7	E_7	E_7	E_8	E_8	F_4	F_4
H	F_4	$\mathrm{Sp}(4)$	$\mathrm{SO}(8)$	$\mathrm{SO}(12)$	$\mathrm{SU}(2)$	E_6	$\mathrm{SO}(2)$	E_7	$\mathrm{SU}(2)$	$\mathrm{Sp}(3)$	$\mathrm{Sp}(1)$
ℓ	2	2	3	1	6	1	6	1	7	1	3

Remark 30 The usefulness of Theorem 16 becomes even more apparent when dealing with homogeneous spaces of exceptional Lie groups G, the definitions of which are more abstract and which are thus more difficult to handle. In Table 1 we collect several examples of triples G, H, ℓ such that G/H admits a locally free homogeneous action of an ℓ-torus $T_\ell \subseteq G$. We recall that the lower index in each E_6, E_7, E_8 and F_4 also indicates the rank thereof [2, p. 40]. (To see that in each of the listed cases H can indeed be viewed as a subgroup of G, see, e.g., [2, Theorem 6.6, pp. 93–94].)

2.5 Other examples related to Theorem 16

The following example shows that the conclusion of Theorem 16 can not be strengthened by claiming the existence of a torus $T_\ell \subseteq G$ with a free homogeneous action $T_\ell \curvearrowright G/H$. The group H considered therein is finite. Similar examples with H connected can be found with the help of Proposition 32 or 34 below.

Example 31 Let G be a compact connected Lie group with rank $n \in \mathbb{N}$, T be a maximal torus in G and $H \subseteq T$ be a nontrivial finite subgroup. Then $\operatorname{rank}(G) - \operatorname{rank}(H) = n$ and for every n-torus $T_n \subseteq G$, the homogeneous action $T_n \curvearrowright G/H$ is locally free by Lemma 9. Now given T_n, we have $gT_ng^{-1} = T$ for some $g \in G$, hence

$$gT_ng^{-1} \cap H = T \cap H = H \neq \{1\}.$$

Lemma 9 thus shows that the action $T_n \curvearrowright G/H$ is not free.

The next two propositions show that free homogeneous actions of tori can be found in some situations. They are of interest also in connection with Theorem 1.

Proposition 32 *Given integers $2 \leq \ell < n$, the following conditions are equivalent:*

(i) there is an ℓ-torus $T_\ell \subseteq \mathrm{U}(n)$ whose homogeneous action on $\mathrm{V}_\ell(\mathbb{C}^n)$ is free,
(ii) $n = \ell + 1$.

Remark 33 The case $\ell = 1$ is included in Example 36.

Proof of Proposition 32 To see that (i) follows from (ii), assume that $n = \ell + 1$ and consider the morphism

$$\varphi\colon \mathbb{T}^\ell \to \mathrm{U}(n), \quad \varphi(x_1, \ldots, x_\ell) = \operatorname{diag}(x_1, \ldots, x_\ell, x_1 \ldots x_\ell).$$

Then φ is a topological isomorphism onto its image, hence $T_\ell = \operatorname{im}(\varphi)$ is an ℓ-torus in $\mathrm{U}(n)$. To show that the homogeneous action of T_ℓ on $\mathrm{V}_\ell(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}(1)$ is free, fix $t \in T_\ell$, write $t = \varphi(x_1, \dots, x_\ell)$ and assume that $gtg^{-1} \in \mathrm{U}(1)$ for some $g \in \mathrm{U}(n)$. This means that 1 is an eigenvalue of gtg^{-1} with the geometric multiplicity at least ℓ, hence the same is true for t. Thus we infer that at least ℓ elements among $x_1, \dots, x_\ell, x_1 \dots x_\ell$ equal 1, which is possible only when $x_i = 1$ for each i. Hence $t = 1$, thus verifying the freeness condition.

To see that (ii) follows from (i), let T_ℓ be an ℓ-torus in $\mathrm{U}(n)$ acting freely on $\mathrm{V}_\ell(\mathbb{C}^n)$. By Lemma 9, this means that

$$gT_\ell g^{-1} \cap \mathrm{U}(n-\ell) = \{1\} \tag{2.3}$$

for every $g \in \mathrm{U}(n)$. In view of Remark 10, we may assume that T_ℓ is contained in the standard maximal torus T of $\mathrm{U}(n)$,

$$T = \{\mathrm{diag}(z_1, \dots, z_n) \colon z_i \in \mathbb{T}^1\}.$$

Fix a topological isomorphism $\psi \colon \mathbb{T}^\ell \to T_\ell$ and write $\psi = \mathrm{diag}(\chi_1, \dots, \chi_n)$. Clearly, $\chi_i \in (\mathbb{T}^\ell)^*$, the Pontryagin dual of $\mathbb{T}^\ell$, for each i. Recall (say, from our proof of Lemma 14) that every permutation of diagonal entries of matrices from T can be realized as a conjugation by some $g \in \mathrm{U}(n)$. Consequently, (2.3) implies that 1 is the only element of T_ℓ having at least ℓ diagonal entries equal to 1. In other words,

($\star$) if $x \in \mathbb{T}^\ell$ and $\chi_i(x) = 1$ for at least ℓ indices i then $\chi_i(x) = 1$ for every i.

Now let $1 \le i_1 < \dots < i_\ell \le n$. Since ψ is a monomorphism, condition ($\star$) shows that the characters $\chi_{i_1}, \dots, \chi_{i_\ell}$ separate points, hence the group they generate is $(\mathbb{T}^\ell)^*$. Thus

($\bullet$) $\chi_{i_1}, \dots, \chi_{i_\ell}$ is a basis of $(\mathbb{T}^\ell)^*$ for each ℓ-tuple $i_1 < \dots < i_\ell$.

We show that this is possible only when $n = \ell + 1$.

So assume, on the contrary, that $n > \ell + 1$ and write

$$\chi_{\ell+1} = \sum_{j=1}^{\ell} k_j \chi_j \quad \text{and} \quad \chi_{\ell+2} = \sum_{j=1}^{\ell} l_j \chi_j. \tag{2.4}$$

Since, for every $j = 1, \dots, \ell$, χ_j can be expressed as an integral combination of the basis $\chi_1, \dots, \chi_{j-1}, \chi_{j+1}, \dots, \chi_\ell, \chi_{\ell+1}$, we infer that $k_j = \pm 1$ (and, similarly, $l_j = \pm 1$) for each $j = 1, \dots, \ell$. By replacing $\chi_{\ell+1}$ with $-\chi_{\ell+1}$ and/or $\chi_{\ell+2}$ with $-\chi_{\ell+2}$, if necessary, we may assume that $k_1 = l_1 = 1$. Now express χ_1 as an integral combination of the basis $\chi_{\ell+1}, \chi_{\ell+2}, \chi_3, \dots, \chi_\ell$,

$$\chi_1 = r\chi_{\ell+1} + s\chi_{\ell+2} + \sum_{j=3}^{\ell} p_j \chi_j. \tag{2.5}$$

By substituting (2.4) into (2.5) and checking the coefficients at χ_1 and χ_2, we obtain $r + s = 1$ and $r \pm s = 0$, a contradiction. □

Proposition 34 *Given integers $1 \leq \ell < n$, the following conditions are equivalent:*

(i) there is a 2-torus $T_2 \subseteq \mathrm{U}(n)$ whose homogeneous action on $\mathrm{V}_\ell(\mathbb{C}^n)$ is free,
(ii) $n \leq 3(\ell - 1)$.

Remark 35 By Theorem 16, (i) implies $\ell \geq 2$, hence also $n \geq 3$.

Proof We show that (i) follows from (ii). To this end, consider the characters χ_1, χ_2, χ_3 of $\mathbb{T}^2$, defined by

$$\chi_1(x, y) = x, \quad \chi_2(x, y) = y, \quad \chi_3(x, y) = xy.$$

Since $3 \leq n \leq 3(\ell - 1)$, we may arrange χ_1, χ_2, χ_3 into an n-tuple $(\varphi_1, \dots, \varphi_n)$ in such a way that each χ_j appears as φ_i for at least one and at most $\ell - 1$ indices i. Define

$$\varphi \colon \mathbb{T}^2 \to \mathrm{U}(n), \quad \varphi = \operatorname{diag}(\varphi_1, \dots, \varphi_n).$$

Since φ is a topological isomorphism onto its image, $T_2 = \operatorname{im}(\varphi)$ is a 2-torus in $\mathrm{U}(n)$. We show that T_2 acts freely on $\mathrm{V}_\ell(\mathbb{C}^n)$. So fix $t = \varphi(x, y) \in T_2$ and $g \in \mathrm{U}(n)$, and assume that $gtg^{-1} \in \mathrm{U}(n - \ell)$. This means that at least ℓ diagonal entries of t equal 1. Thus, by definition of φ, at least two of x, y, xy are 1. This is possible only when $x = y = 1$, hence $t = \varphi(x, y) = 1$.

We show that (ii) follows from (i). Proceeding by contradiction, let $n \geq 3(\ell - 1) + 1$, fix a 2-torus $T_2 \subseteq \mathrm{U}(n)$ along with a topological isomorphism $\psi \colon \mathbb{T}^2 \to T_2$, and assume that the homogeneous action of T_2 on $\mathrm{V}_\ell(\mathbb{C}^n)$ is free. Similarly to our proof of Proposition 32, we shall assume that T_2 is contained in the standard maximal torus T of $\mathrm{U}(n)$. Write $\psi = \operatorname{diag}(\chi_1, \dots, \chi_n)$, notice that $\chi_i \in (\mathbb{T}^2)^*$ and write $\chi_i(x, y) = x^{k_i} y^{l_i}$ ($k_i, l_i \in \mathbb{Z}$). Now divide the set $\{1, \dots, n\}$ into three parts, depending on the parities of the numbers k_i, l_i, as follows:

$$I_1 = \{i : k_i \text{ is even}\}, \; I_2 = \{i : k_i \text{ is odd and } l_i \text{ is even}\}, \; I_3 = \{i : \text{both } k_i, l_i \text{ are odd}\},$$

and consider the points $t_1, t_2, t_3 \in \mathbb{T}^2$ defined by

$$t_1 = (-1, 1), \; t_2 = (1, -1), \; t_3 = (-1, -1).$$

Since $n \geq 3(\ell - 1) + 1$, the Dirichlet's principle yields $j \in \{1, 2, 3\}$ with I_j containing at least ℓ elements. Clearly, $\chi_i(t_j) = 1$ for every $i \in I_j$, hence at least ℓ diagonal entries of $\psi(t_j)$ equal 1. By conjugating with an appropriate matrix $g \in \mathrm{U}(n)$, we obtain $g\psi(t_j)g^{-1} \in \mathrm{U}(n - \ell)$. Since T_2 acts freely on $\mathrm{V}_\ell(\mathbb{C}^n)$, we infer that $\psi(t_j) = 1$, which contradicts our assumption of injectivity of ψ. □

Contrary to the situation of Propositions 32 and 34, a free homogeneous action of a 1-torus is supported by all complex Stiefel manifolds.

Example 36 Fix integers $1 \leq \ell < n$. We have

$$\operatorname{rank}(\mathrm{U}(n)) - \operatorname{rank}(\mathrm{U}(n-\ell)) = \ell \geq 1,$$

hence Theorem 16 yields a locally free homogeneous action of a 1-torus in $\mathrm{U}(n)$ on $\mathrm{V}_\ell(\mathbb{C}^n)$. Let us find a 1-torus which acts freely. Set

$$\varphi\colon \mathbb{T}^1 \to \mathrm{U}(n), \quad \varphi(z) = \operatorname{diag}(z, \dots, z).$$

Then φ is a topological monomorphism, hence $T_1 = \operatorname{im}(\varphi)$ is a 1-torus in $\mathrm{U}(n)$. To see that T_1 acts freely, assume that $t = \varphi(z)$ and $g \in \mathrm{U}(n)$ satisfy $gtg^{-1} \in \mathrm{U}(n-\ell)$. Then 1 is an eigenvalue of gtg^{-1}, hence also of t. Consequently, $z = 1$ and so $t = 1$.

Our last example in this subsection is concerned with the problem of isotropy stability of actions from Theorem 16. It illustrates that the isotropy group may differ from point to point, hence one can not hope for turning the action into a free one by factoring out a "common" isotropy group.

Example 37 Let $G = \mathrm{U}(4)$, $H = \mathrm{U}(2) \subseteq \mathrm{U}(4)$ and T be the standard maximal torus of $\mathrm{U}(4)$. Consider the 2-torus

$$T_2 = \{t_{x,y} = \operatorname{diag}(x^2, y^2, xy, xy^{-1})\colon x, y \in \mathbb{T}^1\} \subseteq T.$$

We show that the homogeneous action of T_2 on G/H is locally free but the isotropy groups $I_g = T_2 \cap gHg^{-1}$ $(g \in G)$ depend on g.

To verify the local freeness fix $g \in G$ and assume that $t_{x,y} \in I_g$. Then $g^{-1}t_{x,y}g \in H$, hence at least two diagonal entries of $t_{x,y}$ equal 1. This is possible only when $x, y \in \{\pm 1\}$, in which case $t_{x,y} = \operatorname{diag}(1, 1, 1, 1)$ or $t_{x,y} = \operatorname{diag}(1, 1, -1, -1)$. Thus I_g is a finite group.

Next we show that $I_1 = 1$. Indeed, if $t_{x,y} \in H$ then $xy = xy^{-1} = 1$, hence also $x^2 = y^2 = 1$, meaning that $t_{x,y} = 1$.

We finish our discussion by finding $g \in G$ with $I_g \neq 1$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{C}^4$ and let g be the operator acting by transpositions $e_1 \leftrightarrow e_3$ and $e_2 \leftrightarrow e_4$. Take $x = 1$ and $y = -1$ so that $t_{x,y} = \operatorname{diag}(1, 1, -1, -1)$. Then $g^{-1}t_{x,y}g = \operatorname{diag}(-1, -1, 1, 1) \in g^{-1}T_2 g \cap H$, hence $t_{x,y} \in T_2 \cap gHg^{-1} = I_g$ and so $I_g \neq 1$ indeed.

2.6 Other examples of locally free actions

In the last part of this section we mention one more natural way of constructing manifolds with locally free actions of tori, namely Proposition 38, and illustrate its application by two examples.

Proposition 38 *Let X be a smooth compact connected manifold without boundary on which $\mathbb{T}^\ell$ $(\ell \geq 1)$ acts smoothly and freely. Let F be a finite group acting on X smoothly, freely and $\mathbb{T}^\ell$-equivariantly. Then X/F admits a smooth locally free action of $\mathbb{T}^\ell$ (hence also a minimal diffeomorphism).*

Remark 39 Let us mention that we let X/F carry the smooth structure that makes the quotient map $p\colon X \to X/F$ a smooth covering map [20, Theorem 9.19, p. 226]. Thus X/F becomes a smooth compact connected manifold without boundary.

Proof By the equivariance assumption, the action of $\mathbb{T}^\ell$ on X descends via p to a continuous action of $\mathbb{T}^\ell$ on X/F, in symbols $tp(x) = p(tx)$ for all $t \in \mathbb{T}^\ell$ and $x \in X$. This descended action is smooth, since p is a smooth covering map. To verify its local freeness, fix $x \in X$ and let $t \in \mathbb{T}^\ell$ be such that $tp(x) = p(x)$, that is, $tx = fx$ for some $f \in F$. Then $f^k = 1$, where k denotes the order of F. By equivariance, $t^k x = f^k x = x$, and since $\mathbb{T}^\ell$ acts freely, $t^k = 1$. Thus t belongs to the finite subgroup of $\mathbb{T}^\ell$ formed by the k-torsions, and the isotropy group of $p(x)$ in $\mathbb{T}^\ell$ is therefore finite. The second statement of the proposition now follows from Theorem 3. □

As an application of Proposition 38, let us look in detail at two examples mentioned in [7, Exemple 3.9].

Example 40 (Lens spaces) Given an integer $n \geq 2$, consider the $(2n-1)$-sphere $\mathbb{S}^{2n-1}$ as an embedded manifold in $\mathbb{C}^n$,

$$\mathbb{S}^{(2n-1)} = \left\{(x_1, \dots, x_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} |x_j|^2 = 1\right\},$$

and let $\mathbb{T}^n$ act on $\mathbb{S}^{2n-1}$ via the diagonal action

$$(z_1, \dots, z_n)(x_1, \dots, x_n) = (z_1 x_1, \dots, z_n x_n).$$

This action is smooth and restricts to a free action of the 1-torus

$$T_1 = \{(z, \dots, z) \colon z \in \mathbb{T}^1\} \subseteq \mathbb{T}^n.$$

Now, given integers $d \geq 2$ and $c_1, \dots, c_n$ with $\gcd(c_j, d) = 1$ for each j, let F be the finite subgroup of $\mathbb{T}^n$ generated by the point

$$f = \left(e^{i2\pi \frac{c_1}{d}}, \dots, e^{i2\pi \frac{c_n}{d}}\right).$$

Clearly, the action of F on $\mathbb{S}^{2n-1}$, obtained by restricting that of $\mathbb{T}^n$, is free and T_1-equivariant by commutativity of $\mathbb{T}^n$. Thus we may use Proposition 38 to see that T_1 acts smoothly and locally freely on the lens space

$$\mathrm{L}(d; c_1, \dots, c_n) = \mathbb{S}^{2n-1}/F.$$

It follows, in particular, that all lens spaces admit minimal diffeomorphisms.

Example 41 Let G be a compact connected Lie group with rank ℓ, T_ℓ be a maximal torus in G and $F \subseteq G$ be a finite group. Let T_ℓ and F act on G by rotations, T_ℓ from the left and F from the right. These two actions are clearly smooth, free and commuting. Thus G/F admits a smooth locally free action of T_ℓ by Proposition 38.

Notice that the previous example fits also into the setting of our Theorem 16.

3 Minimal analytic group extensions

3.1 Cocycles and coboundaries

Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow and G be an abelian topological group. A *cocycle* over $\mathcal{F}$ with values in G is a continuous map $\mathcal{C}\colon \Gamma \times X \to G$ satisfying the identity

$$\mathcal{C}(\alpha, \mathcal{F}(\beta, x)) + \mathcal{C}(\beta, x) = \mathcal{C}(\alpha\beta, x).$$

Such maps form an abelian group $\mathbf{Z}_{\mathcal{F}}(G)$ under the pointwise defined operations. A cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ is called a *coboundary* over $\mathcal{F}$ if it is of the form

$$\mathcal{C}(\gamma, x) = \xi(\mathcal{F}(\gamma, x)) - \xi(x)$$

for some continuous map $\xi : X \to G$. We write $\mathcal{C} = \operatorname{co}(\xi)$ and call ξ a *transfer function* of $\mathcal{C}$. The coboundaries form a subgroup $\mathbf{B}_{\mathcal{F}}(G)$ of $\mathbf{Z}_{\mathcal{F}}(G)$, the corresponding quotient *cohomology group* $\mathbf{Z}_{\mathcal{F}}(G)/\mathbf{B}_{\mathcal{F}}(G)$ will be denoted by $\mathbf{H}_{\mathcal{F}}(G)$. Notice that for every $\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(G)$ and every $z \in X$ there is a unique transfer function ξ of $\mathcal{C}$ with $\xi(z) = 0$.

3.2 Group extensions

Given a minimal flow $\mathcal{F}\colon \Gamma \curvearrowright X$, a torus T and a cocycle $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(T)$, we consider the *group extension* $\mathcal{F}_{\mathcal{C}}$ of $\mathcal{F}$, $\mathcal{F}_{\mathcal{C}}\colon \Gamma \curvearrowright X \times T$, defined by

$$\mathcal{F}_{\mathcal{C}}(\gamma, x, t) = (\mathcal{F}(\gamma, x), \mathcal{C}(\gamma, x) + t).$$

On occasions we shall attribute to $\mathcal{C}$ the dynamical properties of $\mathcal{F}_{\mathcal{C}}$. In particular, we shall say that $\mathcal{C}$ is minimal when the flow $\mathcal{F}_{\mathcal{C}}$ is minimal.

Now let $x \in X$ and denote by $F(\mathcal{C})$ the vertical x-section of the orbit closure $\overline{\mathcal{O}}_{\mathcal{F}_{\mathcal{C}}}(x, 0)$ of the point $(x, 0)$ under the action of $\mathcal{F}_{\mathcal{C}}$. One can show (see, e.g., [5, Section 3.2]) that $F(\mathcal{C})$ is a closed subgroup of T whose definition does not depend on x. The following lemma collects some useful properties of the function F. The reader is referred to [5, Theorem 3.7, p. 63] for a proof.

Lemma 42 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow, T be a torus and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(T)$. Then the following statements hold:*

(a) $\mathcal{C} \in \boldsymbol{B}_{\mathcal{F}}(T)$ if, and only if, $F(\mathcal{C}) = 0$,
(b) $\mathcal{C}$ is minimal if, and only if, $F(\mathcal{C}) = T$,
(c) for every endomorphism q of T, $F(q\mathcal{C}) = q\,(F(\mathcal{C}))$.

For a proof of the following lemma see, e.g., [5, Corollary 3.8, p. 63].

Lemma 43 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow, T be a torus and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(T)$. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ *is minimal,*
(ii) $\chi\mathcal{C}$ *is minimal for every non-zero character* χ *of* T*,*
(iii) $\chi\mathcal{C} \notin \boldsymbol{B}_{\mathcal{F}}(\mathbb{T}^1)$ *for every non-zero character* χ *of* T*.*

3.3 First weak homology groups and induced morphisms

Given a connected manifold X with a fixed base point, we denote the fundamental group of X by $\pi_1(X)$. We shall view the first homology group $H_1(X)$ of X as the abelianization of $\pi_1(X)$. The first weak homology group $H_1^w(X)$ of X is then defined as the quotient group of $H_1(X)$ modulo the torsion subgroup $\mathrm{tor}(H_1(X))$. If X is compact or a Lie group then $\pi_1(X)$ is finitely generated, hence so are $H_1(X)$ and $H_1^w(X)$. In particular, in such case $H_1^w(X)$ is a free abelian group with a finite rank $\mathrm{r}(H_1^w(X))$. Since $\pi_1(\mathbb{T}^1)$ is an abelian torsion-free group, we have isomorphisms

$$\pi_1(\mathbb{T}^1) = H_1(\mathbb{T}^1) = H_1^w(\mathbb{T}^1).$$

If $f : X \to Y$ is a continuous base point preserving map between connected manifolds, we denote by f_* the morphism $H_1^w(X) \to H_1^w(Y)$ induced by f. Notice that f_* descends from the induced morphism $\pi_1(X) \to \pi_1(Y)$ via the quotient morphisms $\pi_1(X) \to H_1^w(X)$ and $\pi_1(Y) \to H_1^w(Y)$.

3.4 Minimal flows with free cycles

Let $\mathcal{F} : \Gamma \curvearrowright X$ be a minimal flow. When searching for a minimal group extension of $\mathcal{F}$ in, say, $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$, we may have no concrete group extensions of $\mathcal{F}$, other than the coboundaries, to begin with. Thus we consider the subgroup $\mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$ of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$ and try to construct the desired minimal extension by applying certain operations to elements of $\mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$. If the group Γ is amenable then one can succeed by using a limit type argument and find minimal extensions of $\mathcal{F}$ in the closure of $\mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$ via the Baire category method. This approach was used in [4, Theorem 8], following earlier ideas of Glasner and Weiss from [9].

When the group Γ is not amenable, the approach used in [4] does not apply and one has to follow different directions, one of which was suggested in [5]. To sketch its idea, consider a coboundary $\mathrm{co}(\xi) \in \mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$ with a transfer function ξ, a prime number d and the d-endomorphism κ_d of $\mathbb{T}^1$, $\kappa_d(t) = dt$. Assume that $\mathrm{co}(\xi)$ lifts across κ_d to a cocycle $\mathcal{C}_d \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$, that is, let $\kappa_d\mathcal{C}_d = \mathrm{co}(\xi)$. Assuming further that ξ does not lift to a continuous map $X \to \mathbb{T}^1$ across κ_d, we may use [5, Lemma 5.1, p. 125] to see that $\mathcal{C}_d$ is not a coboundary, hence $F(\mathcal{C}_d) = \mathbb{Z}_d$ by Lemma 42(a). Now, since the groups $\mathbb{Z}_d$ have a dense union in $\mathbb{T}^1$, an appropriate limit type argument applied to the cocycles $\mathcal{C}_d$ might yield a cocycle $\mathcal{C}$ in $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$ with $F(\mathcal{C}) = \mathbb{T}^1$, which amounts to minimality of $\mathcal{C}$ by Lemma 42(b). In fact, as was shown in [5, Section 3.3], the desired cocycle $\mathcal{C}$ can be obtained as a sum of the cocycles $\mathcal{C}_d$ along a carefully chosen sequence of indices d.

To apply the method described above, we need a continuous map $\xi : X \to \mathbb{T}^1$ fulfilling the following conditions:

- $\mathrm{co}(\xi)$ lifts across κ_d for every $d \geq 2$,
- ξ does not lift across κ_d for any $d \geq 2$.

Both these conditions can be expressed in terms of fundamental groups (or first weak homology groups) and induced morphisms between them (cf. Sect. 3.5 below). It then turns out that such a map ξ exists if, and only if, the flow $\mathcal{F}$ has a free cycle in the following sense.

Definition 44 [5] Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow, $x \in X$, $\mathcal{F}_x\colon \Gamma \to X$, $\gamma \mapsto \mathcal{F}(\gamma, x)$, be the transition map and $(\mathcal{F}_x)_*\colon H_1^w(\Gamma) \to H_1^w(X)$ be the morphism induced by $\mathcal{F}_x$. Set

$$H_1^w(\mathcal{F}) = \mathrm{im}((\mathcal{F}_x)_*).$$

We say that the flow $\mathcal{F}$ *has a free cycle* if

$$\mathrm{r}(H_1^w(\mathcal{F})) < \mathrm{r}(H_1^w(X)).$$

We recall from [5, Remark 2.8, p. 46] that this notion is independent on the choice of a base point x.

3.5 Lifts of cocycles and transfer functions

Given a manifold X with the base point z and an abelian topological group G, we denote by $C_z(X, G)$ the set of all continuous maps $\xi\colon X \to G$ with $\xi(z) = 0$. With operations defined pointwise, $C_z(X, G)$ is an abelian group.

Lemma 45 *Let X be a connected manifold, $z \in X$, $\xi \in C_z(X, \mathbb{T}^1)$ and $p\colon \mathbb{R} \to \mathbb{T}^1$ be the exponential. Then the following conditions are equivalent:*

(i) there is $\eta \in C_z(X, \mathbb{R})$ with $p\eta = \xi$,
(ii) the induced morphism $\xi_*\colon H_1^w(X) \to H_1^w(\mathbb{T}^1)$ vanishes.

If that is the case then such an η is unique.

Proof Since the exponential p is a universal covering, condition (i) occurs precisely when ξ induces the zero morphism on fundamental groups (see, e.g., [23, Lemma 79.1, p. 478]). This is equivalent to condition (ii) on the account of the isomorphism $\pi_1(\mathbb{T}^1) = H_1^w(\mathbb{T}^1)$. □

The following lemma is based on [5, Corollary 5.7, p. 129].

Lemma 46 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow, $z \in X$, $\xi \in C_z(X, \mathbb{T}^1)$ and $p\colon \mathbb{R} \to \mathbb{T}^1$ be the exponential. Denote by ξ_* the morphism $H_1^w(X) \to H_1^w(\mathbb{T}^1)$ induced by ξ. Then the following conditions are equivalent:*

(a) there is $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $p\mathcal{D} = \mathrm{co}(\xi)$,
(b) $\xi_*(H_1^w(\mathcal{F})) = 0$.

If that is the case then such a $\mathcal{D}$ is unique.

Proof Choose $(1, z)$ as the base point for $\Gamma \times X$. By Lemma 45, $\operatorname{co}(\xi)$ lifts across p to a continuous map $\mathcal{D}\colon \Gamma \times X \to \mathbb{R}$ with $\mathcal{D}(1, z) = 0$ if, and only if, the induced morphism $\operatorname{co}(\xi)_*\colon H_1^w(\Gamma \times X) \to H_1^w(\mathbb{T}^1)$ vanishes. Moreover, if that is the case then $\mathcal{D}$ is unique. Now, by [5, Lemma 2.17, p. 55],

$$\operatorname{co}(\xi)_*(H_1^w(\Gamma \times X)) = \xi_*(H_1^w(\mathcal{F})),$$

hence $\operatorname{co}(\xi)_* = 0$ is equivalent to $\xi_*(H_1^w(\mathcal{F})) = 0$. Thus to finish the proof we need only show that if $\mathcal{D}$ exists then it satisfies the cocycle identity. To this end, consider the map

$$\varphi\colon \Gamma^2 \times X \to \mathbb{R}, \quad \varphi(\alpha, \beta, x) = \mathcal{D}(\alpha, \mathcal{F}(\beta, x)) + \mathcal{D}(\beta, x) - \mathcal{D}(\alpha\beta, x).$$

Since $p\mathcal{D} = \operatorname{co}(\xi)$ is a cocycle, $p\varphi = 0$. Moreover, $\varphi(1, 1, z) = \mathcal{D}(1, z) = 0$, hence the uniqueness part of Lemma 45 yields $\varphi = 0$. □

3.6 Morphisms induced by analytic maps

The following lemma is apparently well known in differential topology. Since we are not able to provide an explicit reference, we present its proof for the sake of completeness.

Lemma 47 *Let X be an analytic (respectively, smooth) compact connected manifold without boundary, $z \in X$ and let $h \in \operatorname{Hom}(H_1^w(X), H_1^w(\mathbb{T}^1))$. Then there is an analytic (respectively, smooth) map $\xi \in C_z(X, \mathbb{T}^1)$ with $h = \xi_*$.*

Proof We shall deal with the analytic case, for the smooth case see Remark 48 below.

Let $C(X)$ denote the linear space of all continuous complex-valued functions on X. We equip $C(X)$, as well as its subset $C_z(X, \mathbb{T}^1)$, with the topology of uniform convergence. The supremum norm on $C(X)$ which induces this topology will be denoted by $\|.\|$. Recall that $C_z(X, \mathbb{T}^1)$ is an abelian topological group with the pointwise defined operations derived from the group operations on $\mathbb{T}^1$. Since the map

$$C_z(X, \mathbb{T}^1) \to \operatorname{Hom}(H_1^w(X), H_1^w(\mathbb{T}^1)), \quad \varphi \mapsto \varphi_*,$$

is a morphism of groups vanishing on the open subgroup of $C_z(X, \mathbb{T}^1)$ formed by the null-homotopic maps [5, p. 39], it is locally constant. Now, being a compact manifold, X has the homotopy type of a CW complex [12, Corollary A.12, p. 529], hence we may use [12, Proposition 1B.9, p. 90] to find $\varphi \in C_z(X, \mathbb{T}^1)$ with $h = \varphi_*$. Choose $\varepsilon > 0$ such that $\xi_* = h$ for all $\xi \in C_z(X, \mathbb{T}^1)$ with $\|\xi - \varphi\| < \varepsilon$. By the Grauert–Remmert theorem [14, Theorem 5.1, p. 65], the analytic maps are dense in $C_z(X, \mathbb{T}^1)$, hence there is $\xi \in C_z(X, \mathbb{T}^1)$ analytic with $\|\xi - \varphi\| < \varepsilon$. Then $\xi_* = h$ by our choice of ε. □

Remark 48 The smooth case in Lemma 47 can be proved by following the same line as in the analytic case, using [14, Theorem 2.6, p. 49] in place of the Grauert–Remmert approximation theorem.

3.7 Existence of minimal analytic group extensions

The following theorem is our main result in this section. Though we believe it to be of interest on its own, it will also serve us as one of the tools used to prove Theorem 58 in Sect. 4.

Theorem 49 *Let* $\mathcal{F}\colon \Gamma \curvearrowright X$ *be an analytic (respectively, smooth) minimal flow having a free cycle and let* $\ell \in \mathbb{N}$. *Then* $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^{\ell})$ *contains a subgroup* $\mathbf{R}$ *isomorphic to* $\mathbb{R}$ *such that all* $\mathcal{C} \in \mathbf{R}$ *are analytic (respectively, smooth) and* $\mathcal{F}_{\mathcal{C}}$ *is minimal for every* $0 \neq \mathcal{C} \in \mathbf{R}$.

Remark 50 Results analogous to Theorem 49 hold also in the topological category and with general compact abelian groups G in the fibres, see [5, Sect. 6.1]. Certain ideas from [5] play an important role also in our proof of Theorem 49, but the latter is more constructive and does not involve the algebraic methods used in [5] to handle the general fibre G.

Proof We shall prove the theorem under the analyticity assumption. The smooth case follows by the same argument.

Step 1. We start with some notation and observations.

Let $p\colon \mathbb{R} \to \mathbb{T}^1$ be the exponential. Write

$$n = \mathrm{r}(H_1^w(\mathcal{F})) \quad \text{and} \quad n + m = \mathrm{r}(H_1^w(X))$$

for the ranks of the groups $H_1^w(\mathcal{F})$ and $H_1^w(X)$, respectively. Since $\mathcal{F}$ has a free cycle, $m \geq 1$. Set

$$\mathcal{H} = \left\{ h \in \operatorname{Hom}\left(H_1^w(X), H_1^w(\mathbb{T}^1)\right) : h(H_1^w(\mathcal{F})) = 0 \right\}.$$

Clearly, $\mathcal{H}$ is a free abelian group of rank m. Fix a basis $h_1, \ldots, h_m$ of $\mathcal{H}$.

Choose a base point z for X. By Lemma 47, for every $j = 1, \ldots, m$ there is an analytic map $\xi_j \in C_z(X, \mathbb{T}^1)$ with $h_j = (\xi_j)_*$. Since $\mathcal{F}$ is analytic, it follows that so are all $\mathrm{co}(\xi_j)$. Now

$$(\xi_j)_*(H_1^w(\mathcal{F})) = h_j(H_1^w(\mathcal{F})) = 0,$$

hence Lemma 46 yields $\mathcal{D}_j \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ with $\mathrm{co}(\xi_j) = p\mathcal{D}_j$. Since $\mathcal{D}_j$ is continuous, $\mathrm{co}(\xi_j)$ is analytic and p is an analytic covering map, we infer that $\mathcal{D}_j$ is analytic for every j.

Step 2. Let $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ satisfy $p\mathcal{D} \in \mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$. We show that there are $k_1, \ldots, k_m \in \mathbb{Z}$ and $\eta \in C_z(X, \mathbb{R})$ with

$$\mathcal{D} = \sum_{j=1}^{m} k_j \mathcal{D}_j + \mathrm{co}(\eta).$$

Moreover, such $k_1, \ldots, k_m$ and η are uniquely determined by $\mathcal{D}$.

We begin by verifying the existence part. Let $\xi \in C_z(X, \mathbb{T}^1)$ be the transfer function of $p\mathcal{D}$. By Lemma 46, $\xi_*(H_1^w(\mathcal{F})) = 0$, hence $\xi_* \in \mathcal{H}$. Consequently, there are $k_1, \dots, k_m \in \mathbb{Z}$ such that

$$\xi_* = \sum_{j=1}^{m} k_j h_j = \sum_{j=1}^{m} k_j (\xi_j)_*.$$

Thus

$$\Big(\xi - \sum_{j=1}^{m} k_j \xi_j\Big)_* = 0$$

and so Lemma 45 yields $\eta \in C_z(X, \mathbb{R})$ with $\xi - \sum_{j=1}^{m} k_j \xi_j = p\eta$. It follows that

$$p\Big(\mathcal{D} - \sum_{j=1}^{m} k_j \mathcal{D}_j\Big) = \operatorname{co}(\xi) - \sum_{j=1}^{m} k_j \operatorname{co}(\xi_j) = \operatorname{co}(p\eta) = p \operatorname{co}(\eta),$$

hence by the uniqueness part of Lemma 46,

$$\mathcal{D} = \sum_{j=1}^{m} k_j \mathcal{D}_j + \operatorname{co}(\eta).$$

To verify the uniqueness part, assume that

$$\sum_{j=1}^{m} k_j \mathcal{D}_j + \operatorname{co}(\eta) = \sum_{j=1}^{m} k'_j \mathcal{D}_j + \operatorname{co}(\eta') \tag{3.1}$$

for some $k_j, k'_j \in \mathbb{Z}$ and $\eta, \eta' \in C_z(X, \mathbb{R})$. By applying p we obtain

$$\operatorname{co}\Big(\sum_{j=1}^{m} k_j \xi_j + p\eta\Big) = \operatorname{co}\Big(\sum_{j=1}^{m} k'_j \xi_j + p\eta'\Big).$$

Since the base point preserving transfer functions of coboundaries are unique, we infer that

$$\sum_{j=1}^{m} k_j \xi_j + p\eta = \sum_{j=1}^{m} k'_j \xi_j + p\eta'.$$

Taking now the induced morphisms, we get

$$\sum_{j=1}^{m} k_j h_j = \sum_{j=1}^{m} k_j (\xi_j)_* = \sum_{j=1}^{m} k'_j (\xi_j)_* = \sum_{j=1}^{m} k'_j h_j.$$

Since $h_1, \dots, h_m$ form a basis of $\mathcal{H}$, it follows that $k_j = k_j'$ for every j. With the help of (3.1) this yields $\operatorname{co}(\eta) = \operatorname{co}(\eta')$, hence also $\eta = \eta'$.

STEP 3. We show that

$$S = \left\{s \in \mathbb{R}\colon\ p(s\mathcal{D}_1) \in \mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)\right\}$$

is a free abelian subgroup of $\mathbb{R}$ with rank at most m.

Since $\mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$ is a subgroup of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$, S is a subgroup of $\mathbb{R}$. Given $s \in S$, by Step 2 there are unique $k_1, \dots, k_m \in \mathbb{Z}$ and $\eta \in C_z(X, \mathbb{R})$ with $s\mathcal{D}_1 = \sum_{j=1}^{m} k_j \mathcal{D}_j + \operatorname{co}(\eta)$. The assignment $s \mapsto (k_1, \dots, k_m)$ then defines a morphism of groups $\kappa\colon S \to \mathbb{Z}^m$ and it suffices to verify that κ is a monomorphism. Proceeding by contradiction, let $0 \neq s \in S$ and assume that $\kappa(s) = 0$. Choose $\eta \in C_z(X, \mathbb{R})$ with $s\mathcal{D}_1 = \operatorname{co}(\eta)$. Then $\mathcal{D}_1$ is a coboundary with the transfer function $\vartheta = (1/s)\eta$. Since $p\mathcal{D}_1 = \operatorname{co}(\xi_1)$, it follows that $\xi_1 = p\vartheta$. Thus $(\xi_1)_* = 0$ by Lemma 45, contradicting our choice of ξ_1 from Step 1.

STEP 4. We show that

$$T = \left\{t \in \mathbb{R}\colon\ p(t\mathcal{D}_1) \text{ is not minimal}\right\}$$

is a rational linear subspace of $\mathbb{R}$ with dimension at most m.

In view of Step 3 it suffices to verify the equality

$$T = \left\{\frac{s}{n} : s \in S, n \in \mathbb{N}\right\}.$$

Before turning to the proof, recall from Lemma 42(c) that $F(\kappa_n \mathcal{C}) = \kappa_n\left(F(\mathcal{C})\right)$ for all $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$ and $n \in \mathbb{N}$, where κ_n is the n-endomorphism of $\mathbb{T}^1$.

To verify inclusion "$\subseteq$", fix $t \in T$. By Lemma 42(b), $F(p(t\mathcal{D}_1)) = \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Consequently,

$$F\left(p(nt\mathcal{D}_1)\right) = F(\kappa_n p(t\mathcal{D}_1)) = \kappa_n\left(F(p(t\mathcal{D}_1))\right) = \kappa_n(\mathbb{Z}_n) = 1,$$

whence $p(nt\mathcal{D}_1) \in \mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$ by Lemma 42(a). Thus $nt \in S$.

To verify inclusion"$\supseteq$", choose $s \in S$, $n \in \mathbb{N}$ and write $t = s/n$. Since $p(s\mathcal{D}_1) \in \mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$, we have $F(p(s\mathcal{D}_1)) = 1$, hence

$$\kappa_n\left(F(p(t\mathcal{D}_1))\right) = F(p(nt\mathcal{D}_1)) = F(p(s\mathcal{D}_1)) = 1.$$

Consequently, $F(p(t\mathcal{D}_1)) \subseteq \mathbb{Z}_n$, and it follows from Lemma 42(b) that $p(t\mathcal{D}_1)$ is not minimal. Thus $t \in T$.

STEP 5. It follows from Step 4 that

$$\mathbb{R} = T \oplus R$$

for some rational linear subspace R of $\mathbb{R}$ with dimension c. Given $r = (r_1, \dots, r_\ell) \in R^\ell$, set

$$\mathcal{C}_r = (p(r_1\mathcal{D}_1), p(r_2\mathcal{D}_1), \dots, p(r_\ell\mathcal{D}_1)) \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^\ell).$$

Since both p and $\mathcal{D}_1$ are analytic, so is $\mathcal{C}_r$. We show that if $r_1, \dots, r_\ell$ are rationally independent then $\mathcal{C}_r$ is minimal.

We employ Lemma 43. Fix a non-zero character χ of $\mathbb{T}^\ell$. Then there are $k_1, \dots, k_\ell \in \mathbb{Z}$, not all of them zero, such that

$$\chi(x_1, \dots, x_\ell) = \sum_{i=1}^{\ell} k_i x_i$$

for every $(x_1, \dots, x_\ell) \in \mathbb{T}^\ell$. Setting $\varrho = \sum_{i=1}^{\ell} k_i r_i$, we obtain

$$\chi\mathcal{C}_r = \sum_{i=1}^{\ell} k_i p(r_i\mathcal{D}_1) = p(\varrho\mathcal{D}_1).$$

Since $r_1, \dots, r_\ell$ are rationally independent, $0 \neq \varrho \in R$, hence $\varrho \notin T$ by our choice of R. Thus $\chi\mathcal{C}_r = p(\varrho\mathcal{D}_1)$ is minimal by definition of T, and so the minimality of $\mathcal{C}_r$ follows from Lemma 43.

STEP 6. Fix a rational linear basis of R and group its elements into ℓ-tuples, letting them form a subset $\mathcal{B}$ of R^ℓ with cardinality c. With operations defined coordinate-wise, R^ℓ is a rational linear space with dimension c. Denote by $\mathcal{R}$ the subspace of R^ℓ spanned by $\mathcal{B}$. We show that the elements of $\mathcal{B}$ are independent over $\mathbb{Q}$, hence $\mathcal{B}$ is a basis of $\mathcal{R}$. It will follow that $\mathcal{R}$ is isomorphic to $\mathbb{R}$.

To verify the independence of elements of $\mathcal{B}$, fix $k \in \mathbb{N}$ along with mutually distinct points

$$r^{(i)} = \left(r_1^{(i)}, \dots, r_\ell^{(i)}\right) \in \mathcal{B} \tag{3.2}$$

for $i = 1, \dots, k$. Let further $q_1, \dots, q_k \in \mathbb{Q}$ be such that $\sum_{i=1}^{k} q_i r^{(i)} = 0$. By taking into account the first coordinate, we infer that $\sum_{i=1}^{k} q_i r_1^{(i)} = 0$. Since $r_1^{(i)}$ $(i = 1, \dots, k)$ are rationally independent by our definition of $\mathcal{B}$, it follows that $q_i = 0$ for every i.

STEP 7. It follows from Step 5 that $\mathcal{C}_r$ is analytic for every $r \in R^\ell$ and minimal for every $r \in \mathcal{B}$. Consequently, $\mathcal{C}_u$ is analytic for every $u \in \mathcal{R}$ and we verify that it is minimal for every $0 \neq u \in \mathcal{R}$.

So let $0 \neq u \in \mathcal{R}$. Choose $k \in \mathbb{N}$, distinct points $r^{(1)}, \dots, r^{(k)} \in \mathcal{B}$ and rationals $q_1, \dots, q_k$, not all of them zero, such that $u = \sum_{i=1}^{k} q_i r^{(i)}$. To verify the minimality of $\mathcal{C}_u$, by Step 5 it suffices to show that the ℓ coordinates $u_1, \dots, u_\ell$ of u are rationally independent. So let $q'_1, \dots, q'_\ell \in \mathbb{Q}$ and assume that $\sum_{j=1}^{\ell} q'_j u_j = 0$. Keeping notation (3.2) from Step 6, we obtain

$$0=\sum_{j=1}^{\ell} q'_j \sum_{i=1}^{k} q_i r_j^{(i)} = \sum_{j=1}^{\ell}\sum_{i=1}^{k}\left(q'_j q_i\right) r_j^{(i)}.$$

Now, since all $r_j^{(i)}$ are rationally independent, it follows that $q'_j q_i = 0$ for all j and i. By considering i with $q_i \neq 0$, we conclude with $q'_j = 0$ for $j = 1, \dots, \ell$.

STEP 8. Summarizing our discussion so far, $\mathcal{R}$ is a group isomorphic to $\mathbb{R}$, $\mathcal{C}_u \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^\ell)$ is analytic for every $u \in \mathcal{R}$ and minimal for $0 \neq u \in \mathcal{R}$. We finish the proof by showing that the assignment

$$\mathcal{R} \to \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^\ell), \quad u \mapsto \mathcal{C}_u,$$

is a monomorphism of groups.

The fact that this is a morphism of groups follows by an elementary verification. As for injectivity, if $0 \neq u \in \mathcal{R}$ then $\mathcal{C}_u$ is minimal, hence non-zero. □

Remark 51 By composing the monomorphism $\mathcal{R} \to \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^\ell)$ from Step 8 of our proof of Theorem 49 with the quotient morphism $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^\ell) \to \mathbf{H}_{\mathcal{F}}(\mathbb{T}^\ell)$, we obtain a morphism $\mathcal{R} \to \mathbf{H}_{\mathcal{F}}(\mathbb{T}^\ell)$ which is monic, since $\mathcal{C}_u \notin \mathbf{B}_{\mathcal{F}}(\mathbb{T}^\ell)$ for every $0 \neq u \in \mathcal{R}$. Thus we have an isomorphic copy of $\mathbb{R}$ within $\mathbf{H}_{\mathcal{F}}(\mathbb{T}^\ell)$, whose non-zero elements are represented by minimal and analytic extensions of $\mathcal{F}$.

4 Minimal smooth skew products

4.1 Almost free actions of the 2-torus

This subsection contains auxiliary results concerning almost free actions of $\mathbb{T}^2$, of which Proposition 55 will be particularly useful to us later in our proof of Theorem 58. We also discuss Examples 56 and 57 mentioned in Remark 7, Sect. 1.

Lemma 52 *Let Z be a compact metrizable space and let $\mathbb{T}^2$ act locally freely on Z. Then the isotropy groups $I_z \subseteq \mathbb{T}^2$ ($z \in Z$) are of uniformly bounded order. The subgroup of $\mathbb{T}^2$ generated by their union is finite, hence so is also the set $\{I_z : z \in Z\}$.*

Before proving Lemma 52 let us recall several facts concerning the hyperspace of $\mathbb{T}^2$. The nonempty closed subsets of $\mathbb{T}^2$ form a compact metrizable space $\mathcal{T}$ with the Hausdorff-Vietoris topology [16, Theorem 3.5, p. 18]. By compactness of $\mathbb{T}^2$, a set $T \in \mathcal{T}$ is a subgroup of $\mathbb{T}^2$ if, and only if, $TT \subseteq T$. Consequently, the closed subgroups of $\mathbb{T}^2$ form a closed, hence compact subspace of $\mathcal{T}$.

Proof of Lemma 52 To verify the first assertion assume, on the contrary, that $(z_n)_{n=1}^{\infty}$ is a sequence in Z with $\operatorname{ord}(I_{z_n}) \to \infty$ as $n \to \infty$. Since Z is compact, we may assume that $z_n \to z$ for some $z \in Z$. By passing to a suitable subsequence, if necessary, we find a closed subgroup J of $\mathbb{T}^2$ with $I_{z_n} \to J$ in the Hausdorff-Vietoris topology. Clearly, J is infinite and $J \subseteq I_z$, contradicting the local freeness assumption.

Now let d be the smallest common multiplier of the orders of the groups I_z. Since $\mathbb{Z}_d \oplus \mathbb{Z}_d$ is the kernel of the d-endomorphism of $\mathbb{T}^2$, we have $I_z \subseteq \mathbb{Z}_d \oplus \mathbb{Z}_d$ for every z and the second statement of the lemma thus follows. □

Every 1-torus in $\mathbb{T}^2$ can be written in the form

$$T_{k,l} = \{(z^k, z^l) \colon z \in \mathbb{T}^1\}$$

for a pair of integers k, l, not both of them zero. Clearly, we may additionally assume that $\gcd(k, l) = 1$, in which case

$$T_{k,l} = \{(x, y) \colon x^l = y^k\}.$$

Every finite cyclic subgroup of $\mathbb{T}^2$ is of the form

$$F^d_{m,n} = \left\langle \left(e^{i2\pi \frac{m}{d}}, e^{i2\pi \frac{n}{d}}\right)\right\rangle,$$

where m, n are integers, d is a positive integer and $\gcd(m, n, d) = 1$.

Lemma 53 *Let k, l be integers, not both of them zero. Let m, n be integers and d be a positive integer with* $\gcd(m, n, d) = 1$. *Consider the following statements:*

(a) $T_{k,l} \cap F^d_{m,n} = \{1\}$,
(b) $\gcd(d, ml - nk) = 1$.

Then (a) *follows from* (b) *and the converse holds if* $\gcd(k, l) = 1$.

Proof To verify the first implication, assume that $\gcd(d, ml - nk) = 1$ and fix $z \in T_{k,l} \cap F^d_{m,n}$. Choose $j \in \mathbb{N}$ with

$$z = \left(e^{i2\pi \frac{mj}{d}}, e^{i2\pi \frac{nj}{d}}\right).$$

Since $z \in T_{k,l}$, $j(ml - nk)/d$ is an integer, hence d divides j by (b). Thus $z = 1$.

To verify the converse implication, assume that $\gcd(k, l) = 1$ and $T_{k,l} \cap F^d_{m,n} = \{1\}$, and fix a positive integer p dividing both d and $ml - nk$. Set $j = d/p$ and define

$$z = (x, y) = \left(e^{i2\pi \frac{mj}{d}}, e^{i2\pi \frac{nj}{d}}\right) = \left(e^{i2\pi \frac{m}{p}}, e^{i2\pi \frac{n}{p}}\right).$$

Then $z \in F^d_{m,n}$. Moreover, $(ml - nk)/p$ is an integer, hence $x^l = y^k$. Since k, l are coprime, we infer that $z \in T_{k,l}$. Thus $z = 1$ by (a), hence p divides both m and n. Since $\gcd(m, n, d) = 1$, we conclude with $p = 1$. □

Lemma 54 *Let Z be a compact metrizable space and let $\mathbb{T}^2$ act on Z almost freely. Then each isotropy group $I_z \subseteq \mathbb{T}^2$ ($z \in Z$) is cyclic.*

Proof We shall consider the Pontryagin dual $(\mathbb{T}^2)^*$ of $\mathbb{T}^2$ and identify it, in the usual way, with $\mathbb{Z}^2$. For every closed subgroup F of $\mathbb{T}^2$, $F^\perp$ will denote the annihilator of F in $(\mathbb{T}^2)^*$.

Fix $z \in Z$ and write $I_z = I$. By assumptions, I is a finite group, hence so is also $I^* = \mathbb{Z}^2/I^\perp$ and it follows that $I^\perp$ is a subgroup of $\mathbb{Z}^2$ of rank 2. By the elementary divisors theorem [22, Theorem 4.2, p. 24], there exist an automorphism A of $\mathbb{Z}^2$ and positive integers d, d' with d dividing d', such that $I^\perp = A(d\mathbb{Z} \oplus d'\mathbb{Z})$. Write B for the automorphism of $\mathbb{T}^2$ with $A = B^*$. Then

$$B(I)^\perp = A^{-1}(I^\perp) = d\mathbb{Z} \oplus d'\mathbb{Z} = (\mathbb{Z}_d \oplus \mathbb{Z}_{d'})^\perp,$$

hence $B(I) = \mathbb{Z}_d \oplus \mathbb{Z}_{d'}$. To finish the proof, it suffices to show that $d = 1$.

Let κ_d be the d-endomorphism of $\mathbb{T}^2$. Since $B(I) \supseteq \mathbb{Z}_d \oplus \mathbb{Z}_d = \ker(\kappa_d)$ and B commutes with κ_d, we have $I \supseteq \ker(\kappa_d)$. Now fix k, l coprime with $T_{k,l}$ acting freely on Z and set

$$w = (x, y) = \left(e^{i2\pi \frac{k}{d}}, e^{i2\pi \frac{l}{d}}\right) \in T_{k,l}.$$

Since $w \in \ker(\kappa_d) \subseteq I$, the freeness of $T_{k,l}$ yields $w = 1$. Thus d divides both k and l, and since $\gcd(k, l) = 1$, we have $d = 1$ indeed. □

Proposition 55 *Let Z be a compact metrizable space on which $\mathbb{T}^2$ acts locally freely. Then the following conditions are equivalent:*

(1) the action of $\mathbb{T}^2$ on Z is almost free,
(2) the set

$$Q = \{l/k \colon k \neq 0 \text{ and } T_{k,l} \text{ acts freely on } Z\}$$

is dense in $\mathbb{R}$.

Proof Implication (2)⇒(1) is clear. To verify implication (1)⇒(2), fix a 1-torus $T_1 \subseteq \mathbb{T}^2$ acting freely on Z and write $T_1 = T_{\kappa,\lambda}$ with $\kappa, \lambda \in \mathbb{Z}$ coprime. By Lemma 54, the isotropy group I_z of each $z \in Z$ under the action of $\mathbb{T}^2$ is cyclic, hence can be written in the form $I_z = F^d_{m,n}$ with $\gcd(m, n, d) = 1$. Since $T_{\kappa,\lambda} \cap I_z = \{1\}$ by assumption, for such m, n, d we have

$$\gcd(d, m\lambda - n\kappa) = 1 \tag{4.1}$$

by Lemma 53.

Since the isotropy groups of $\mathbb{T}^2$ form a finite set by Lemma 52, we may define c as the smallest common multiplier of all the numbers d from above and consider the pairs (k, l) of the form

$$k = \kappa + ac, \quad l = \lambda + bc$$

with $a, b \in \mathbb{Z}$ and $k \neq 0$. Clearly, the corresponding fractions l/k are dense in $\mathbb{R}$. To see that $T_{k,l}$ acts freely on Z for each such pair (k, l), fix $z \in Z$ and write $I_z = F^d_{m,n}$ with $\gcd(m, n, d) = 1$. Since d divides c, we have, by virtue of (4.1),

$$\gcd(d, ml - nk) = \gcd(d, (m\lambda - n\kappa) + c(mb - na)) = \gcd(d, m\lambda - n\kappa) = 1.$$

Thus we infer from Lemma 53 that $T_{k,l} \cap I_z = \{1\}$. □

The following two examples show, as indicated in Remark 7, that an almost free action of a 2-torus need not be free and a locally free action need not be almost free. Both examples deal with homogeneous actions of 2-tori on the Stiefel manifold $\mathrm{V}_2(\mathbb{C}^4)$.

Example 56 To show that an almost free action may fail to be free, let us follow the setting and notation from Example 37. As was shown therein, the homogeneous action of T_2 on G/H is locally free. Moreover, since there are isotropy groups that do not vanish, the action is not free. To see that the action is almost free, it suffices to find a 1-torus $T_1 \subseteq T_2$ acting freely. So let

$$T_1 = \{t_{x,y} \in T_2 \colon x = y\} = \{\mathrm{diag}(x^2, x^2, x^2, 1) \colon x \in \mathbb{T}^1\}.$$

If $x \in \mathbb{T}^1$ and $g \in G$ satisfy $g^{-1} t_{x,x} g \in H$, then at least two diagonal entries of $t_{x,x}$ equal 1. This is possible only when $x^2 = 1$, hence $t_{x,x} = 1$. Thus T_1 acts freely by Lemma 9 and so T_2 acts almost freely.

Example 57 We show that a locally free action of a 2-torus need not be almost free. Let $G = \mathrm{U}(4)$, $H = \mathrm{U}(2) \subseteq \mathrm{U}(4)$ and $e_1, \ldots, e_4$ be the standard basis of $\mathbb{C}^4$. Consider the morphism $\varphi \colon \mathbb{T}^2 \to G$, given by

$$\varphi(x, y) = \mathrm{diag}(x^2, y^2, xy^2, x^2 y).$$

Clearly, φ is a topological isomorphism onto its image, hence $T_2 = \mathrm{im}(\varphi)$ is a 2-torus in G. One verifies that two diagonal entries of $\varphi(x, y)$ can equal 1 only for finitely many pairs (x, y), hence the homogeneous action of T_2 on G/H is locally free. Further, if $x, y \in \mathbb{Z}_2$ then the first two diagonal entries of $\varphi(x, y)$ are 1. Thus if $g \in G$ is the operator acting by transpositions $e_1 \leftrightarrow e_3$ and $e_2 \leftrightarrow e_4$ then the isotropy group $I_g = T_2 \cap gHg^{-1}$ of gH contains $\varphi(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, hence is not cyclic. In view of Lemma 54 this means that our action of T_2 on G/H is not almost free.

4.2 The weak C^∞ topology

Let X, Y be smooth manifolds without boundary. We shall denote by $C^\infty(X, Y)$ the set of all smooth maps $f \colon X \to Y$ and equip this set with the weak C^∞ topology [14, Sect. 2.1] (meaning that maps and their derivatives are compared uniformly on compact sets). The subset of $C^\infty(X, X)$ formed by the diffeomorphisms will be denoted by $\mathcal{D}(X)$. We recall from [14, Sect. 2.4] that $C^\infty(X, Y)$ is a completely metrizable space.

Moreover, if X, Y, Z are smooth manifolds without boundary then the composition $(f, g) \mapsto gf$ is continuous when considered as a map

$$C^\infty(X, Y) \times C^\infty(Y, Z) \to C^\infty(X, Z),$$

[14, p. 64]. It follows, in particular, that for all $g \in \mathcal{D}(X)$ and $h \in \mathcal{D}(Y)$, the assignment $f \mapsto hfg$ defines a homeomorphism on $C^\infty(X, Y)$.

4.3 Existence of minimal smooth skew products

Let X, Z be compact connected manifolds without boundary and $\mathcal{F}\colon \Gamma \curvearrowright X, \mathcal{G}\colon \Gamma \curvearrowright X \times Z$ be flows. Recall that $\mathcal{G}$ is called a skew product over $\mathcal{F}$ if it acts as $\mathcal{F}$ on the first coordinate.

The following theorem is our main result in this section. Its proof is based on ideas from [7].

Theorem 58 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a smooth minimal flow with a free cycle and let Z be a smooth compact connected manifold without boundary admitting a smooth almost free action of $\mathbb{T}^2$. Then there is a smooth minimal flow $\Gamma \curvearrowright X \times Z$ in the form of a skew product over $\mathcal{F}$.*

Proof We shall proceed in nine steps.

Step i. We invoke several objects from our proof of Theorem 49 and recall some of their properties.

Let T_γ $(\gamma \in \Gamma)$ be the acting diffeomorphisms of $\mathcal{F}$. Denote the action of $\mathbb{T}^2$ on Z by ϕ and let φ_t $(t \in \mathbb{T}^2)$ be the underlying acting diffeomorphisms. Consider the exponential $p\colon \mathbb{R} \to \mathbb{T}^1$. We have a smooth cocycle $\mathcal{D}_1 \in \mathbf{Z}_{\mathcal{F}}(\mathbb{R})$ and a rational linear subspace R of $\mathbb{R}$ with dimension c such that

$$\mathcal{C}_r = (p(r_1\mathcal{D}_1), p(r_2\mathcal{D}_1)) \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^2)$$

is minimal (that is, the group extension $\mathcal{F}_{\mathcal{C}_r}$ is minimal) for every $r = (r_1, r_2) \in R^2$ with r_1, r_2 rationally independent. We shall view R^2 as a topological subspace of $\mathbb{R}^2$. Since R has dimension $c > 1$, the rationally independent pairs $r \in R^2$ form a dense subset of R^2.

It follows from the cocycle identity that

$$\mathcal{D}_1(\alpha^{-1}, \mathcal{F}(\alpha, x)) = -\mathcal{D}_1(\alpha, x)$$

for all $\alpha \in \Gamma$ and $x \in X$, hence $\operatorname{im}(\mathcal{D}_1)$ is a symmetric subset of $\mathbb{R}$. Since $\mathcal{C}_r$ is minimal for each r from a dense subset of $\mathbb{R}^2$, we infer that $\mathcal{D}_1$ is unbounded both from above and below, and since its domain $\Gamma \times X$ is connected, $\operatorname{im}(\mathcal{D}_1) = \mathbb{R}$.

Given $r \in R^2$, we shall consider the smooth flow $\mathcal{F}^r\colon \Gamma \curvearrowright X \times Z$ defined by

$$\mathcal{F}^r(\gamma, x, z) = (\mathcal{F}(\gamma, x), \phi(\mathcal{C}_r(\gamma, x), z))$$

$$\begin{tikzcd}
\Gamma \overset{\mathcal{F}}{\curvearrowright} X \arrow[dd, "\mathcal{C}_r"] & & \\
 & {} \arrow[r] & \mathcal{F}^r\colon \Gamma \curvearrowright X \times Z \\
\mathbb{T}^2 \overset{\phi}{\curvearrowright} Z & &
\end{tikzcd}$$

Fig. 1 The definition of $\mathcal{F}^r$

(see Fig. 1). We shall denote by T^r_γ $(\gamma \in \Gamma)$ the acting diffeomorphisms of $\mathcal{F}^r$. Thus

$$T^r_\gamma(x, z) = (T_\gamma x, \varphi_{\mathcal{C}_r(\gamma,x)} z).$$

Step II. We show that if $r_1, r_2 \in R \setminus \{0\}$ are rationally dependent then $\mathrm{im}(\mathcal{C}_r)$ is a 1-torus in $\mathbb{T}^2$. More precisely, if $lr_1 = kr_2$ with $k, l \in \mathbb{Z}\setminus\{0\}$ then

$$\mathrm{im}(\mathcal{C}_r) = T_{k,l} = \left\{\left(z^k, z^l\right) \colon z \in \mathbb{T}^1\right\}.$$

Indeed, since $\mathrm{im}(\mathcal{D}_1) = \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{im}(\mathcal{C}_r) &= \{(p(r_1 t), p(r_2 t)) \colon t \in \mathbb{R}\} = \{(p(r_1 k t), p(r_2 k t)) \colon t \in \mathbb{R}\} \\
&= \left\{\left(p(r_1 t)^k, p(r_1 t)^l\right) \colon t \in \mathbb{R}\right\} = T_{k,l}.
\end{aligned}$$

Step III. We fix some more notation.

Let

$$\mathcal{X} = C^\infty(\Gamma \times X \times Z, X \times Z).$$

Recall from Sect. 4.2 that $\mathcal{X}$ is a completely metrizable space with the weak C^∞ topology. Given a diffeomorphism $h \in \mathcal{D}(Z)$ and $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2) \in \mathcal{X}$, we shall identify h with $\mathrm{id}_X \times h \in \mathcal{D}(X \times Z)$ and with $\mathrm{id}_{\Gamma\times X} \times h \in \mathcal{D}(\Gamma \times X \times Z)$, and we shall consider the map $h^{-1}\mathcal{E}h \in \mathcal{X}$,

$$h^{-1}\mathcal{E}h(\gamma, x, z) = (\mathcal{E}_1(\gamma, x, h(z)), h^{-1}\mathcal{E}_2(\gamma, x, h(z))).$$

It follows from our discussion in Sect. 4.2 that the assignment $C_h \colon \mathcal{E} \to h^{-1}\mathcal{E}h$ is a homeomorphism on $\mathcal{X}$. In particular, if $r \in R^2$ then $h^{-1}\mathcal{F}^r h \colon \Gamma \curvearrowright X \times Z$ is a smooth flow and its acting diffeomorphisms are $h^{-1}T^r_\gamma h$ $(\gamma \in \Gamma)$.

Set

$$\mathcal{Y} = \{h^{-1}\mathcal{F}^r h \colon r \in R^2, h \in \mathcal{D}(Z)\}$$

and denote by $\mathcal{Z}$ the closure of $\mathcal{Y}$ in $\mathcal{X}$. Being a closed subspace of a completely metrizable space $\mathcal{X}$, $\mathcal{Z}$ is itself completely metrizable. Moreover, since $\mathcal{Y}$ consists of smooth skew products over $\mathcal{F}$, it follows that so does $\mathcal{Z}$. Notice also that the assignment $r \mapsto \mathcal{F}^r$ defines a continuous map $R^2 \to \mathcal{Y}$.

STEP IV. Fix $x \in X$. Given nonempty open sets $U \subseteq X$ and $V \subseteq Z$, denote by $\mathcal{Z}_{U,V}$ the set of all $\mathcal{G} \in \mathcal{Z}$ such that the fibre $\{x\} \times Z$ is covered by the images of $U \times V$ under the action of $\mathcal{G}$. Let $r_1, r_2 \in R$ be rationally independent, $V \subseteq Z$ be open and let $h \in \mathcal{D}(Z)$ be such that $h(V)$ intersects each orbit of the action ϕ of $\mathbb{T}^2$ on Z. We show that $\mathcal{F}^r \in \mathcal{Z}_{U,h(V)}$ for every U.

Since $h(V)$ intersects each orbit of ϕ, the images of $h(V)$ under the action of $\mathbb{T}^2$ cover Z. Now $h(V)$ is open and Z is compact, hence there is a finite set $F \subseteq \mathbb{T}^2$ with $Z = \bigcup_{t \in F} \varphi_t(h(V))$. Choose compact sets $C_t \subseteq \varphi_t(h(V))$ with $Z = \bigcup_{t \in F} C_t$. For every $t \in F$, the condition $\varphi_s(h(V)) \supseteq C_t$ is open with respect to $s \in \mathbb{T}^2$, hence there is a neighbourhood W_t of t in $\mathbb{T}^2$ such that $\varphi_s(h(V)) \supseteq C_t$ for every $s \in W_t$. Choose an identity neighbourhood W in $\mathbb{T}^2$ and open sets $t \in W_t' \subseteq \mathbb{T}^2$ with $W_t' W^{-1} \subseteq W_t$ for every $t \in F$.

Since the flow $\mathcal{F}_{\mathcal{C}_r} : \Gamma \curvearrowright X \times \mathbb{T}^2$ is minimal by Step i, we have

$$\mathcal{F}_{\mathcal{C}_r}(\Gamma \times U \times W) = X \times \mathbb{T}^2 \supseteq \{x\} \times W_t'$$

for every $t \in F$. Thus we can find $\gamma_t \in \Gamma$, $x_t \in U$ and $p_t \in W$ with $T_{\gamma_t} x_t = x$ and $\mathcal{C}_r(\gamma_t, x_t) p_t \in W_t'$. Then $s = \mathcal{C}_r(\gamma_t, x_t) \in W_t' p_t^{-1} \subseteq W_t' W^{-1} \subseteq W_t$, hence

$$T^r_{\gamma_t}(U \times h(V)) \supseteq T^r_{\gamma_t}(\{x_t\} \times h(V)) = \{x\} \times \varphi_s(h(V)) \supseteq \{x\} \times C_t$$

by our choice of W_t. It follows that

$$\bigcup_{t \in F} T^r_{\gamma_t}(U \times h(V)) \supseteq \bigcup_{t \in F} \{x\} \times C_t = \{x\} \times Z,$$

thus verifying $\mathcal{F}^r \in \mathcal{Z}_{U,h(V)}$.

STEP V. As noticed in Step iii, the conjugating maps C_h $(h \in \mathcal{D}(Z))$ are homeomorphisms on $\mathcal{X}$. Since $C_h(\mathcal{Y}) = \mathcal{Y}$, each C_h restricts to a homeomorphism on $\mathcal{Z} = \overline{\mathcal{Y}}$. Fix $\mathcal{G} \in \mathcal{Z}$, $h \in \mathcal{D}(Z)$ and nonempty open sets $U \subseteq X$, $V \subseteq Z$. We show that $h^{-1}\mathcal{G}h \in \mathcal{Z}_{U,V}$ if, and only if, $\mathcal{G} \in \mathcal{Z}_{U,h(V)}$, and, moreover,

$$C_{h^{-1}}\left(\overline{\mathcal{Z}_{U,V}}\right) = \overline{\mathcal{Z}_{U,h(V)}}$$

with the closures taken in $\mathcal{Z}$.

Let S_γ $(\gamma \in \Gamma)$ be the acting diffeomorphisms of $\mathcal{G}$. The first claim follows from the equality

$$\bigcup_{\gamma \in \Gamma} (h^{-1} S_\gamma h)(U \times V) = h^{-1}\Big(\bigcup_{\gamma \in \Gamma} S_\gamma(U \times h(V))\Big)$$

and from the invariance of $\{x\} \times Z$ under the action of h. It can be rewritten as

$$C_{h^{-1}}(\mathcal{Z}_{U,V}) = \mathcal{Z}_{U,h(V)}.$$

Since $C_{h^{-1}}$ is a homeomorphism on $\mathcal{Z}$, the second claim follows.

STEP VI. Given $r = (r_1, r_2) \in R^2$ with r_1, r_2 rationally dependent and nonzero, recall from Step ii that $\operatorname{im}(\mathcal{C}_r)$ is a 1-torus in $\mathbb{T}^2$. Denote by $R_f^{(2)}$ the set of all such pairs r with the action of $\operatorname{im}(\mathcal{C}_r)$ on Z free. We show that $R_f^{(2)}$ is dense in R^2.

Fix $\varrho_1, \varrho_2 \in R \setminus \{0\}$. By Proposition 55, $\varrho = (\varrho_1, \varrho_2)$ can be approximated by pairs of the form $r = (\varrho_1, l\varrho_1/k)$, where $k, l \in \mathbb{Z} \setminus \{0\}$ and $T_{k,l}$ acts freely on Z. For such a pair r we have $\operatorname{im}(\mathcal{C}_r) = T_{k,l}$ by Step ii, hence $r \in R_f^{(2)}$.

STEP VII. We show that $\mathcal{Z}_{U,V}$ is dense in $\mathcal{Z}$ for all nonempty open sets $U \subseteq X$ and $V \subseteq Z$.

Since $\mathcal{Z} = \overline{\mathcal{Y}}$, it suffices to verify the density of $\mathcal{Z}_{U,V}$ in $\mathcal{Y}$. By definition of $\mathcal{Y}$, this amounts to showing that $h^{-1}\mathcal{F}^r h \in \overline{\mathcal{Z}_{U,V}}$ for all $r \in R^2$ and $h \in \mathcal{D}(Z)$. It follows from Step v that this is equivalent to $\mathcal{F}^r \in \overline{\mathcal{Z}_{U,h(V)}}$. Moreover, since $R_f^{(2)}$ is dense in R^2 by Step vi and $\mathcal{F}^r$ depends continuously on r by Step iii, our problem translates to showing that $\mathcal{F}^r \in \overline{\mathcal{Z}_{U,V}}$ for every $r \in R_f^{(2)}$ and all nonempty open sets $U \subseteq X$, $V \subseteq Z$.

So fix such r, U and V. Since $T_1 = \operatorname{im}(\mathcal{C}_r)$ is a 1-torus in $\mathbb{T}^2$ acting freely on Z, we may use [7, Lemme 5.10] to find $h \in \mathcal{D}(Z)$ such that

(a) h commutes with φ_t for every $t \in T_1$, and
(b) $h(V)$ intersects each orbit of the action of $\mathbb{T}^2$.

Condition (a) implies $h^{-1}\mathcal{F}^r h = \mathcal{F}^r$. Condition (b), together with our Steps iv and v, shows that for every rationally independent pair $\varrho \in R^2$, $\mathcal{F}^\varrho \in \mathcal{Z}_{U,h(V)}$, hence $h^{-1}\mathcal{F}^\varrho h \in \mathcal{Z}_{U,V}$. By choosing a sequence of rationally independent pairs $\varrho_n \in R^2$ with $\varrho_n \to r$, we obtain

$$h^{-1}\mathcal{F}^{\varrho_n} h \to h^{-1}\mathcal{F}^r h = \mathcal{F}^r$$

and $h^{-1}\mathcal{F}^{\varrho_n} h \in \mathcal{Z}_{U,V}$, hence $\mathcal{F}^r \in \overline{\mathcal{Z}_{U,V}}$ indeed.

STEP VIII. Denote by $\mathcal{M}$ the intersection of all the sets $\mathcal{Z}_{U,V}$. We show that each $\mathcal{G} \in \mathcal{M}$ is minimal.

Given $\mathcal{G} \in \mathcal{M}$ and nonempty open sets $U \subseteq X$, $V \subseteq Z$, we need to show that the images of $U \times V$ under the action of $\mathcal{G}$ cover $X \times Z$. Since $\mathcal{G} \in \mathcal{Z}_{U,V}$, we have $\mathcal{G}(\Gamma \times U \times V) \supseteq \{x\} \times Z$. By compactness of Z, $\mathcal{G}(\Gamma \times U \times V) \supseteq N \times Z$ for some neighbourhood N of x in X. Further, by minimality of $\mathcal{F}$, the images of N under the action of $\mathcal{F}$ cover X. Since $\mathcal{G}$ acts as $\mathcal{F}$ on the first coordinate, we conclude with

$$\mathcal{G}(\Gamma \times U \times V) \supseteq \mathcal{G}(\Gamma \times N \times Z) \supseteq \mathcal{F}(\Gamma \times N) \times Z = X \times Z.$$

STEP IX. We show that $\mathcal{M} \neq \emptyset$. Since each $\mathcal{G} \in \mathcal{M}$ is a minimal smooth skew product on $X \times Z$ over $\mathcal{F}$, this will finish the proof.

The intersection $\mathcal{M} = \bigcap_{U,V} \mathcal{Z}_{U,V}$ can be expressed as a countable one by restricting to basic sets $U \subseteq X$ and $V \subseteq Z$. Since $\mathcal{Z}$ is a completely metrizable space, in view of the Baire category theorem we need only verify that each $\mathcal{Z}_{U,V}$ is open and dense in $\mathcal{Z}$. The openness holds in the compact-open topology, hence also in the finer topology of $\mathcal{Z}$. The density has been verified in Step vii. □

4.4 Examples of homogeneous almost free actions of the 2-torus

Our aim in this subsection is to find examples of homogeneous spaces admitting homogeneous almost free actions of 2-tori. Our motivation lies in connection with both Theorem 8 and Theorem 58.

Let G be a compact connected Lie group and H be a closed subgroup of G. By Theorem 16, a necessary and sufficient condition for the existence of a 2-torus $T_2 \subseteq G$ with the homogeneous action on G/H locally free is

$$\operatorname{rank}(G) - \operatorname{rank}(H) \geq 2. \tag{4.2}$$

Thus by definition of almost freeness, in order to have a 2-torus in G with an almost free homogeneous action on G/H, condition (4.2) is necessary. Examples 60, 61 and 62 below demonstrate that the condition is not sufficient.

Lemma 59 *Let G be a compact connected Lie group, T be a maximal torus in G and H be a closed subgroup of G. Assume that there is an integer $d \geq 2$ such that*

$$\ker(\kappa_d) \subseteq \bigcup_{g \in G} gHg^{-1},$$

where κ_d denotes the d-endomorphism of T. Then there is no 1-torus in G with a free homogeneous action on G/H.

Proof Fix a 1-torus T_1 in G. In view of Remark 10 we may assume, without loss of generality, that $T_1 \subseteq T$. Choose $1 \neq t \in T_1$ with $t^d = 1$. Then $t \in \ker(\kappa_d)$, hence $t \in gHg^{-1}$ for some $g \in G$. Thus $T_1 \cap gHg^{-1} \neq \{1\}$ and Lemma 9(ii) applies. □

Example 60 Let $n, n_1, \ldots, n_k$ be positive integers with $n = \sum_{i=1}^{k} n_i$, $G = \mathrm{SO}(n)$ and $H = \mathrm{S}(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k))$. Write $m = \lfloor n/2 \rfloor$, consider the usual isomorphism $R\colon \mathbb{T}^1 \to \mathrm{SO}(2)$ and let

$$T = \{\operatorname{diag}(R(z_1), \ldots, R(z_m), 1)\colon z_i \in \mathbb{T}^1\}$$

(with the entry 1 omitted if n is even) be the standard maximal torus of G. Then

$$\ker(\kappa_2) = \left\{ \operatorname{diag}(x_1, x_1, \ldots, x_{2m}, x_{2m}, 1)\colon x_i \in \{\pm 1\} \right\} \subseteq H.$$

By Lemma 59, it follows that there is no 1-torus in G with a free homogeneous action on $G/H = \mathrm{F}_{n_1,\ldots,n_k}(\mathbb{R})$. Notice that (4.2) holds if, and only if, n_i is odd for at least four indices i.

Example 61 The following two examples also fit into the setting of Lemma 59 on the account of $\ker(\kappa_2) \subseteq H$:

(1) $G = \mathrm{U}(n)$, $H = \mathrm{O}(n)$ and

$$T = \{\operatorname{diag}(z_1, \ldots, z_n)\colon z_i \in \mathbb{T}^1\};$$

(2) $G = \mathrm{SU}(n)$, $H = \mathrm{SO}(n)$ and

$$T = \{\mathrm{diag}(z_1, \dots, z_n) \colon z_i \in \mathbb{T}^1, z_1 \dots z_n = 1\}.$$

Thus neither $\mathrm{U}(n)/\,\mathrm{O}(n)$ nor $\mathrm{SU}(n)/\,\mathrm{SO}(n)$ admits an almost free homogeneous action of a 2-torus, even though (4.2) holds for $n \geq 3$ in case (1) and for $n \geq 5$ in case (2).

Lemma 59 also applies to the following situation.

Example 62 Given $n \in \mathbb{N}$, the symplectic group $H = \mathrm{Sp}(n)$ can be identified with the subgroup of $G = \mathrm{SU}(2n)$ formed by the unitary matrices of the form

$$M = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix}$$

with complex $n \times n$ matrices A, B and their complex conjugates A^*, B^*. Notice that

$$\mathrm{rank}(G) - \mathrm{rank}(H) = 2n - 1 - n = n - 1 \geq 2$$

for $n \geq 3$. We show, however, that there is no 2-torus in G with an almost free homogeneous action on G/H.

Consider the standard maximal torus T of G,

$$T = \{\mathrm{diag}(z_1, \dots, z_{2n}) \colon z_1, \dots, z_{2n} \in \mathbb{T}^1, z_1 \dots z_{2n} = 1\}$$

and its 2-endomorphism κ_2. We verify that $\ker(\kappa_2) \subseteq \bigcup_{g \in G} gHg^{-1}$ and then apply Lemma 59. Fix $t \in \ker(\kappa_2)$ and write $t = \mathrm{diag}(x_1, \dots, x_{2n})$. Then $x_i = \pm 1$ for every i and since $\det(t) = 1$, $x_i = 1$ for an even number of indices i. Consequently, $\mathrm{diag}(x_{\sigma(1)}, \dots, x_{\sigma(2n)}) \in H$ for an appropriate permutation σ. Clearly, we may assume, without loss of generality, that σ is even. Now let $e_1, \dots, e_{2n}$ be the standard basis of $\mathbb{C}^{2n}$ and consider the unitary operator k on $\mathbb{C}^{2n}$ acting by $k(e_i) = e_{\sigma(i)}$ for every i. Since σ is even, $\det(k) = 1$, hence $k \in \mathrm{SU}(2n) = G$. Finally,

$$k^{-1}tk = \mathrm{diag}(x_{\sigma(1)}, \dots, x_{\sigma(2n)}) \in H,$$

hence $t \in kHk^{-1}$.

Lemma 63 *Let G be a compact Lie group, H be a closed subgroup of G and $T_1 \subseteq G$ be a 1-torus. Assume that $\omega \colon G \to \mathbb{T}^1$ is a topological morphism restricting to an isomorphism between T_1 and $\mathbb{T}^1$. If $\omega(H) = \{1\}$ then the homogeneous action of T_1 on G/H is free.*

Proof We need to show that $T_1 \cap gHg^{-1} = \{1\}$ for every $g \in G$. So assume that $g \in G$ and $h \in H$ are such that $t = ghg^{-1} \in T_1$. Then

$$\omega(t) = \omega(g)\omega(h)\omega(g)^{-1} = \omega(h) = 1,$$

hence $t = 1$ by injectivity of ω on T_1. □

Proposition 64 *Let $\ell \geq 3$ be an integer and H be a closed subgroup of $\mathrm{SU}(\ell)$. Assume that k is a conjugate linear automorphism of $\mathbb{C}^{\ell}$ that commutes with each $h \in H$. Then there is a 2-torus $T_2 \subseteq \mathrm{U}(\ell)$ whose homogeneous action on $\mathrm{U}(\ell)/H$ is almost free.*

Proof We proceed in three short steps.

Step 1. Let $h \in H$ and $v \in \mathbb{C}^{\ell}$ be an eigenvector of h with the eigenvalue λ. We show that $k(v)$ is an eigenvector of h with the eigenvalue λ^*. Since k is injective, it follows that every imaginary eigenvalue of h has a conjugate counterpart with the same geometric multiplicity.

By applying k to the equality $h(v) = \lambda v$ we obtain

$$hk(v) = kh(v) = k(\lambda v) = \lambda^* k(v).$$

Step 2. Consider the 1-torus $T_1 \subseteq \mathrm{U}(\ell)$ defined by

$$T_1 = \{\mathrm{diag}(x, 1, \ldots, 1) \colon x \in \mathbb{T}^1\}.$$

We show that the homogeneous action of T_1 on $\mathrm{U}(\ell)/H$ is free.

Since $H \subseteq \mathrm{SU}(\ell)$ by assumption, it suffices to apply Lemma 63 to $\omega = \det$.

Step 3. Consider the 2-torus $T_2 \subseteq \mathrm{U}(\ell)$ defined by

$$T_2 = \{t_{x,y} = \mathrm{diag}(x, y, y, 1, \ldots, 1) \colon x, y \in \mathbb{T}^1\}.$$

We show that the homogeneous action of T_2 on $\mathrm{U}(\ell)/H$ is locally free. Since $T_1 \subseteq T_2$, this will finish the proof.

Fix $g \in \mathrm{U}(\ell)$ and assume that $t_{x,y} \in T_2 \cap gHg^{-1}$. Clearly, the eigenvalues of $h = g^{-1}t_{x,y}g \in H$ are precisely the diagonal entries of $t_{x,y}$. In view of Step 1, it follows that $x, y \in \{\pm 1\}$, meaning that $T_2 \cap gHg^{-1}$ is finite. □

Example 65 The following two examples fit into the setting of Proposition 64:

(1) $\ell = n \geq 3$, $H = \mathrm{SO}(n)$ and $k(v) = v^*$ for $v \in \mathbb{C}^n$;
(2) $\ell = 2n$ with $n \geq 2$, $H = \mathrm{Sp}(n)$ (viewed as a closed subgroup of $\mathrm{SU}(2n)$ as indicated in Example 62) and $k(u, v) = (v^*, -u^*)$ for $u, v \in \mathbb{C}^n$.

Thus both $\mathrm{U}(n)/\mathrm{SO}(n)$ $(n \geq 3)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$ $(n \geq 2)$ admit an almost free homogeneous action of a 2-torus.

Lemma 66 *Let $n \geq 2$ be an integer. Then*

$$\mathcal{T}_2 = \{\tau_{x,y} = (x, x^2y^{-1}, x^3y^{-2}, \ldots, x^ny^{-(n-1)}) \colon x, y \in \mathbb{T}^1\}$$

is a 2-torus in $\mathbb{T}^n$ and

$$\mathcal{T}_1 = \{\tau_x = (x, x, \ldots, x) \colon x \in \mathbb{T}^1\}$$

is a 1-torus in $\mathcal{T}_2$. Moreover, if $x, y \in \mathbb{T}^1$ are such that at least two coordinates of $\tau_{x,y}$ equal 1, then $x, y \in \mathbb{Z}_m$, where m is the smallest common multiplier of the numbers $1, \ldots, n-1$.

Proof The first two statements are immediate. To verify the third statement, fix $x, y \in \mathbb{T}^1$ and assume that

$$x^i y^{-(i-1)} = x^{i+d} y^{-(i+d-1)} = 1$$

for some $1 \leq i < i + d \leq n$. Then

$$(xy^{-1})^d = x^{i+d} y^{-(i+d-1)} \big(x^i y^{-(i-1)}\big)^{-1} = 1,$$

hence $xy^{-1} \in \mathbb{Z}_d$. Consequently,

$$x = x^i y^{-(i-1)} \big(xy^{-1}\big)^{-(i-1)} = \big(xy^{-1}\big)^{-(i-1)} \in \mathbb{Z}_d$$

and so

$$y = x\big(xy^{-1}\big)^{-1} \in \mathbb{Z}_d.$$

Since d divides m, we conclude with $x, y \in \mathbb{Z}_m$. □

Proposition 67 *Let $\ell < n$ be positive integers. Then $\mathrm{V}_\ell(\mathbb{C}^n)$ admits an almost free homogeneous action of a 2-torus if, and only if, $\ell \geq 2$.*

Proof The necessity of $\ell \geq 2$ follows from Theorem 16. To verify the sufficiency, write $G = \mathrm{U}(n)$, $H = \mathrm{U}(n - \ell)$ and let m be the smallest common multiplier of the numbers $1, \ldots, n - 1$. Keeping the notation from Lemma 66, consider the 2-torus

$$T_2 = \{\mathrm{diag}(\tau) \colon \tau \in \mathcal{T}_2\} \subseteq G$$

and the 1-torus

$$T_1 = \{\mathrm{diag}(\tau) \colon \tau \in \mathcal{T}_1\} \subseteq T_2.$$

We show that the homogeneous action of T_2 (respectively, of T_1) on $G/H = \mathrm{V}_\ell(\mathbb{C}^n)$ is locally free (respectively, free).

Fix $g \in G$. Assuming that $t = \mathrm{diag}(\tau_{x,y}) \in T_2 \cap gHg^{-1}$, we have $g^{-1}tg \in H$, hence at least two diagonal entries of t equal 1. In view of Lemma 66 this implies $x, y \in \mathbb{Z}_m$, meaning that $T_2 \cap gHg^{-1}$ is finite. Similarly, if $t' = \mathrm{diag}(\tau_x) \in T_1 \cap gHg^{-1}$ then $x = 1$, hence $t' = 1$. Thus $T_1 \cap gHg^{-1} = \{1\}$. □

Remark 68 An argument similar to that from Proposition 67 applies to the quaternionic Stiefel manifolds $\mathrm{V}_\ell(\mathbb{H}^n)$ with $2 \leq \ell < n$.

In what follows we shall view elements of $\mathrm{SO}(n)$ both as matrices and as linear operators on $\mathbb{R}^n$.

Proposition 69 *Let n be a positive integer and H be a closed subgroup of $\mathrm{SO}(n)$. Assume that each $h \in H$ has 1 as an eigenvalue with the geometric multiplicity at least four. Then there is a 2-torus $T_2 \subseteq \mathrm{SO}(n)$ whose homogeneous action on $\mathrm{SO}(n)/H$ is almost free.*

Remark 70 If H is connected and S is its maximal torus then every element of H is conjugate in H to an element of S. Consequently, if 1 is an eigenvalue with the geometric multiplicity at least four for each $s \in S$ then the same is true for each $h \in H$.

Proof of Proposition 69 We proceed in three steps.

STEP 1. We fix some notation.

Write $m = \lfloor n/2 \rfloor$, consider the usual isomorphism $R\colon \mathbb{T}^1 \to \mathrm{SO}(2)$ and let $\varphi\colon \mathbb{T}^m \to \mathrm{SO}(n)$ be defined by

$$\varphi(z_1, \dots, z_m) = \operatorname{diag}(R(z_1), \dots, R(z_m), 1),$$

with the entry 1 omitted if n is even. Then φ is a topological isomorphism onto the standard maximal torus of $\mathrm{SO}(n)$. Following the notation introduced in Lemma 66, consider the 2-torus

$$T_2 = \varphi(\mathcal{T}_2) \subseteq \mathrm{SO}(n)$$

and the 1-torus

$$T_1 = \varphi(\mathcal{T}_1) \subseteq T_2,$$

and denote by m the smallest common multiplier of the numbers $1, \dots, n-1$.

STEP 2. We show that the homogeneous action of T_2 on $\mathrm{SO}(n)/H$ is locally free.

Fix $g \in \mathrm{SO}(n)$, let $t = \varphi(\tau_{x,y}) \in T_2 \cap gHg^{-1}$ and write $\tau_{x,y} = (z_1, \dots, z_m)$. Then $g^{-1}tg \in H$, hence t fixes at least four linearly independent vectors from $\mathbb{R}^n$. By considering the nonzero coordinates of these fixed vectors, we see that there are at least two indices $i \in \{1, \dots, m\}$ such that $R(z_i)$ fixes a non-zero vector from $\mathbb{R}^2$. For these indices i we have $z_i = 1$, hence, by Lemma 66, $x, y \in \mathbb{Z}_m$.

STEP 3. We show that the homogeneous action of T_1 on $\mathrm{SO}(n)/H$ is free.

The proof is similar to that from Step 2. If $t' = \varphi(\tau_x) \in T_1 \cap gHg^{-1}$ then $x = 1$, hence $t' = 1$. □

Example 71 Proposition 69 (and Remark 70) apply to the following situations:

(1) $H = \mathrm{SO}(n - \ell)$ with $\lfloor n/2 \rfloor - \lfloor (n-\ell)/2 \rfloor \geq 2$, here $\mathrm{SO}(n)/H = \mathrm{V}_\ell(\mathbb{R}^n)$;

(2) $H = \mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_k)$ with $n = \sum_{i=1}^k n_i$ and with n_i odd for at least four indices i.

It follows from Theorem 16 that to have a 2-torus with an almost free homogeneous action, the assumptions on ℓ and $n_1, \dots, n_k$ above are necessary.

Remark 72 Given a compact connected Lie group G, its closed subgroups $H \supseteq H'$ and a 2-torus $T_2 \subseteq G$, consider the homogeneous actions $T_2 \curvearrowright G/H$ and $T_2 \curvearrowright G/H'$. By Lemma 9, the isotropy groups of the first action contain those of the second action, so the latter is locally free provided that the former is. Similarly, if $T_1 \subseteq T_2$ is a 1-torus then T_1 acts freely on G/H' as soon as it acts freely on G/H. In summary, if T_2 acts almost freely on G/H then it does so also on G/H'. Thus, for example, it follows from Example 65 that for every $n \geq 3$ there is a 2-torus in $\mathrm{U}(n)$ whose homogeneous action on $\mathrm{U}(n)/\,\mathrm{SO}(n-\ell) \times \mathrm{SO}(\ell)$ is almost free for every $1 \leq \ell < n$.

Remark 73 Other examples of homogeneous spaces admitting homogeneous almost free actions of 2-tori can be found by applying [7, 3.9. Exemple f] to direct products of spaces considered so far. More precisely, let G, G' be compact connected Lie groups with closed subgroups $H \subseteq G$, $H' \subseteq G'$, let a 1-torus $T_1 \subseteq G$ act freely on G/H and a 1-torus $T_1' \subseteq G'$ act locally freely on G'/H'. Then $T_1 \times T_1'$ is a 2-torus in $G \times G'$ whose homogeneous action on $G/H \times G'/H' = (G \times G')/(H \times H')$ is almost free. Thus, for example, it follows from Example 36 and Theorem 16 that $\mathrm{V}_\ell(\mathbb{C}^n) \times G'/H'$ admits an almost free homogeneous action of a 2-torus if $1 \leq \ell < n$ and $\operatorname{rank}(G') - \operatorname{rank}(H') \geq 1$.

We conclude this section with the following problem.

Problem 74 Table 1 contains examples of homogeneous spaces of exceptional Lie groups which admit locally free homogeneous actions of tori T_ℓ with various dimensions $\ell \geq 2$. Does any of these spaces admit an almost free homogeneous action of a 2-torus?

5 Free cycles in homogeneous flows

5.1 Remarks on notation

Given a connected manifold X with a fixed base point, we shall consider the fundamental group $\pi_1(X)$, the first homology group $H_1(X)$ and the first weak homology group $H_1^w(X)$ of X, respectively. Recall that $H_1(X)$ is the abelianization of $\pi_1(X)$ and $H_1^w(X)$ is the quotient group of $H_1(X)$ modulo the torsion subgroup. If X is compact or a Lie group then $\pi_1(X)$ is finitely generated, hence so are both $H_1(X)$ and $H_1^w(X)$, and $H_1^w(X)$ is the torsion-free part of $H_1(X)$. In the case when X is a Lie group we shall always use the identity of X as its base point; recall that in this case $\pi_1(X)$ is abelian.

If $f\colon X \to Y$ is a continuous base point preserving map between connected manifolds X, Y, we consider the induced morphisms

$$f_*\colon \pi_1(X) \to \pi_1(Y), \quad f_*^{ab}\colon H_1(X) \to H_1(Y) \text{ and } f_*^w\colon H_1^w(X) \to H_1^w(Y).$$

Notice that this notation differs from the one we used in Sect. 3, where, for the sake of simplicity, f_* was used to denote the induced morphism between the first weak homology groups.

Given a group Ω, write Ω^{ab} for the abelianization of Ω and Ω^w for the quotient group of Ω^{ab} modulo its torsion subgroup. The (torsion-free) rank of a finitely generated abelian group A is denoted by $\mathrm{r}(A)$. If Λ is a subgroup of a group Γ then we let Γ/Λ consist of the left cosets of Γ modulo Λ. In the case when Γ is a Lie group and Λ is its closed subgroup we assume that Γ/Λ is carrying the structure of a smooth manifold making the canonical projection $p\colon \Gamma \to \Gamma/\Lambda$ a submersion.

5.2 Free cycles in transitive homogeneous flows

Our interest in this section is in the existence of free cycles in homogeneous flows. We discuss transitive actions first, postponing the general homogeneous flows to Sect. 5.6.

Theorem 75 *Let Γ be a connected Lie group, Λ be its closed cocompact subgroup and $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$ be the homogeneous flow. Then*

$$\mathrm{r}(H_1^w(\Gamma/\Lambda)) - \mathrm{r}(H_1^w(\mathcal{F})) = \mathrm{r}((\Lambda/\Lambda_0)^w),$$

where Λ_0 denotes the identity component of Λ. In particular, $\mathcal{F}$ has a free cycle if, and only if,

$$(\Lambda/\Lambda_0)^w \neq 0.$$

Proof We divide the proof into four steps.

Step 1. We collect several preliminary observations and fix some notation.

Choose $z = \Lambda$ as the base point for Γ/Λ so that the transition map $\mathcal{F}_z\colon \Gamma \to \Gamma/\Lambda$ coincides with the canonical quotient map p. Being a bundle projection, p has the lifting property of paths and their homotopies [21, Lemma 4.2, p. 86]. Given a loop $f \in \pi_1(\Gamma/\Lambda)$, consider the endpoint $\lambda \in \Lambda$ of a lift $\widetilde{f}$ of f across p starting at 1. While λ may depend on the choice of $\widetilde{f}$, the coset $\lambda\Lambda_0 \in \Lambda/\Lambda_0$ does not and the assignment $f \mapsto \lambda^{-1}\Lambda_0 = (\lambda\Lambda_0)^{-1}$ defines a morphism of groups $r\colon \pi_1(\Gamma/\Lambda) \to \Lambda/\Lambda_0$. Clearly, r is an epimorphism and its kernel coincides with the image of the morphism $p_*\colon \pi_1(\Gamma) \to \pi_1(\Gamma/\Lambda)$ induced by p. Thus we have an exact sequence

$$\pi_1(\Gamma) \xrightarrow{p_*} \pi_1(\Gamma/\Lambda) \xrightarrow{r} \Lambda/\Lambda_0 \longrightarrow 0. \tag{5.1}$$

Notice that (5.1) fits into the long exact sequence of homotopy groups of the bundle projection p [12, Theorem 4.41, p. 376]. To simplify notation in the rest of the proof, put $\Omega = \Lambda/\Lambda_0$.

Consider the abelianizations $H_1(\Gamma)$, $H_1(\Gamma/\Lambda)$, Ω^{ab} of $\pi_1(\Gamma)$, $\pi_1(\Gamma/\Lambda)$, Ω, respectively, and denote by q_1, q_2, q_3 the underlying quotient morphisms. Recall that the kernel of each q_i is the commutator subgroup of its domain. Clearly, p_* and r project via the morphisms q_i to morphisms $p_*^{ab}\colon H_1(\Gamma) \to H_1(\Gamma/\Lambda)$ and $r^{ab}\colon H_1(\Gamma/\Lambda) \to \Omega^{ab}$, respectively. Since $\pi_1(\Gamma)$ is abelian, q_1 is an isomorphism. The commutator subgroups of $\pi_1(\Gamma/\Lambda)$ and Ω will be denoted by $\pi_1(\Gamma/\Lambda)'$ and Ω', respectively.

Consider the quotient groups $H_1^w(\Gamma)$, $H_1^w(\Gamma/\Lambda)$, Ω^w of $H_1(\Gamma)$, $H_1(\Gamma/\Lambda)$, Ω^{ab} modulo the torsion subgroups and let r_1, r_2, r_3 be the corresponding quotient morphisms. Then p_*^{ab} and r^{ab} descend via the morphisms r_i to morphisms $p_*^w\colon H_1^w(\Gamma) \to H_1^w(\Gamma/\Lambda)$ and $r^w\colon H_1^w(\Gamma/\Lambda) \to \Omega^w$, respectively. Thus we have a commutative diagram depicted in Fig. 2.

Step 2. We show that the second row in Fig. 2 is an exact sequence. (We shall keep the additive notation throughout the rest of the proof even though $\pi_1(\Gamma/\Lambda)$ and Ω may fail to be abelian.)

Fig. 2 A commutative diagram associated with (5.1)

$$\begin{tikzcd}
\pi_1(\Gamma) \arrow[r, "p_*"] \arrow[d, "q_1"] & \pi_1(\Gamma/\Lambda) \arrow[r, "r"] \arrow[d, "q_2"] & \Omega \arrow[r] \arrow[d, "q_3"] & 0 \\
H_1(\Gamma) \arrow[r, "p_*^{ab}"] \arrow[d, "r_1"] & H_1(\Gamma/\Lambda) \arrow[r, "r^{ab}"] \arrow[d, "r_2"] & \Omega^{ab} \arrow[r] \arrow[d, "r_3"] & 0 \\
H_1^w(\Gamma) \arrow[r, "p_*^w"] & H_1^w(\Gamma/\Lambda) \arrow[r, "r^w"] & \Omega^w \arrow[r] & 0
\end{tikzcd}$$

Fig. 3 Towards the exactness at $H_1(\Gamma/\Lambda)$

$$\begin{tikzcd}
 & f' \arrow[r, "r"] & \omega \arrow[d, equal] \\
g \arrow[d, "q_1"] & f \arrow[r, "r"] \arrow[d, "q_2"] & \omega \arrow[d, "q_3"] \\
k \arrow[r, "p_*^{ab}"] & h \arrow[r, "r^{ab}"] & 0
\end{tikzcd}$$

First, since r is an epimorphism, so is r^{ab}, whence follows exactness at Ω^{ab}. Further, $rp_* = 0$ yields $r^{ab}p_*^{ab} = 0$, hence $\operatorname{im}(p_*^{ab}) \subseteq \ker(r^{ab})$. To show that $\operatorname{im}(p_*^{ab}) \supseteq \ker(r^{ab})$, we use a diagram chasing argument illustrated in Fig. 3. Fix $h \in \ker(r^{ab})$, choose $f \in \pi_1(\Gamma/\Lambda)$ with $q_2(f) = h$ and write $\omega = r(f)$. Then $q_3(\omega) = r^{ab}(h) = 0$, hence $\omega \in \Omega'$. Since r is an epimorphism, there is $f' \in \pi_1(\Gamma/\Lambda)'$ with $r(f') = \omega$. Now $f - f' \in \ker(r) = \operatorname{im}(p_*)$, hence $f - f' = p_*(g)$ for some $g \in \pi_1(\Gamma)$. Put $k = q_1(g)$. Since $q_2(f') = 0$,

$$p_*^{ab}(k) = q_2 p_*(g) = q_2(f - f') = q_2(f) = h,$$

which shows that $h \in \operatorname{im}(p_*^{ab})$ indeed.

Step 3. The third row in Fig. 2 is exact at Ω^w, since r^w is an epimorphism. We show that

$$\operatorname{im}(p_*^w) \subseteq \ker(r^w) \quad \text{and} \quad d\ker(r^w) \subseteq \operatorname{im}(p_*^w), \tag{5.2}$$

where d is the largest torsion coefficient of Ω^{ab}. (Notice that Ω, being an epimorphic image of a finitely generated group $\pi_1(\Gamma/\Lambda)$, is itself finitely generated. It follows that so is Ω^{ab}, hence d is well defined.)

The first inclusion in (5.2) follows from $r^{ab}p_*^{ab} = 0$. We verify the second inclusion by a diagram chasing argument illustrated in Fig. 4. Fix $s \in \ker(r^w)$, choose $f \in H_1(\Gamma/\Lambda)$ with $r_2(f) = s$ and put $\kappa = r^{ab}(f)$. Since $r_3(\kappa) = 0$, $\kappa \in \operatorname{tor}(\Omega^{ab})$, hence $d\kappa = 0$. Consequently, $df \in \ker(r^{ab}) = \operatorname{im}(p_*^{ab})$, whence $df = p_*^{ab}(g)$ for some $g \in H_1(\Gamma)$. Letting $k = r_1(g)$, we obtain

$$ds = r_2 p_*^{ab}(g) = p_*^w r_1(g) = p_*^w(k),$$

Fig. 4 Towards the inclusion $d\ker(r^w) \subseteq \mathrm{im}(p^w_*)$

$$\begin{tikzcd}
g \arrow[r, "p^{ab}_*"] \arrow[d, "r_1"] & df \arrow[d, "r_2"] & f \arrow[r, "r^{ab}"] \arrow[d, "r_2"] & \kappa \arrow[d, "r_3"] \\
k \arrow[r, "p^w_*"] & ds & s \arrow[r, "r^w"] & 0
\end{tikzcd}$$

meaning that $ds \in \mathrm{im}(p^w_*)$ indeed.

STEP 4. We prove the theorem.

First notice that all the groups $H^w_1(\Gamma)$, $H^w_1(\Gamma/\Lambda)$, Ω^w are free abelian with finite ranks. Inclusions (5.2) imply that $\ker(r^w)/\mathrm{im}(p^w_*)$ is a torsion group with the largest torsion coefficient dividing d. Consequently, the free abelian groups $\ker(r^w)$ and $\mathrm{im}(p^w_*)$ have the same rank. Further, since Ω^w is free abelian and r^w is an epimorphism, Ω^w can be viewed as a direct summand in $H^w_1(\Gamma/\Lambda)$ with the complementary summand $\ker(r^w)$. Hence

$$\mathrm{r}(H^w_1(\Gamma/\Lambda)) = \mathrm{r}(\Omega^w) + \mathrm{r}(\ker(r^w)) = \mathrm{r}(\Omega^w) + \mathrm{r}(\mathrm{im}(p^w_*)).$$

Finally, since $\mathrm{im}(p^w_*) = H^w_1(\mathcal{F})$, we obtain

$$\mathrm{r}(\Omega^w) = \mathrm{r}(H^w_1(\Gamma/\Lambda)) - \mathrm{r}(\mathrm{im}(p^w_*)) = \mathrm{r}(H^w_1(\Gamma/\Lambda)) - \mathrm{r}(H^w_1(\mathcal{F})).$$

□

Existence of free cycles has the following topological consequences for Λ and Γ.

Corollary 76 *Let Γ be a connected Lie group, Λ be its closed cocompact subgroup and $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$ be the homogeneous flow. If $\mathcal{F}$ has a free cycle then Λ/Λ_0 is infinite (hence Λ is disconnected) and Γ is non-compact.*

Proof By Theorem 75, $(\Lambda/\Lambda_0)^w$ is a non-trivial torsion-free abelian group, hence is infinite. Consequently, Λ/Λ_0 is also infinite. Since Λ_0 is open in Λ, it follows that Λ is non-compact, hence so is Γ. □

5.3 Homogeneous flows induced by covering maps

The following immediate corollary of Theorem 75 deals with the case when Λ is a uniform lattice (that is, a discrete cocompact subgroup) of Γ.

Proposition 77 *Let Γ be a connected Lie group and $\Lambda \subseteq \Gamma$ be a uniform lattice. Then the homogeneous flow $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$ has a free cycle if, and only if, $\Lambda^w \neq 0$.*

Example 78 Given $x, y, z \in \mathbb{R}$, write

$$[x, y, z] = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the Heisenberg group

$$\Gamma = \mathrm{H}_3(\mathbb{R}) = \{[x, y, z] \colon x, y, z \in \mathbb{R}\}$$

and its uniform lattice

$$\Lambda = \mathrm{H}_3(\mathbb{Z}) = \{[x, y, z] \colon x, y, z \in \mathbb{Z}\}.$$

The commutator subgroup of Λ is

$$\Lambda' = \{[0, 0, z] \colon z \in \mathbb{Z}\}$$

and the abelianization $\Lambda^{ab} = \Lambda/\Lambda'$ of Λ is thus isomorphic to $\mathbb{Z}^2$. It follows that $\mathrm{r}(\Lambda^w) = 2$, hence the homogeneous flow $\mathcal{F} \colon \Gamma \curvearrowright \Gamma/\Lambda$ has a free cycle by Proposition 77.

Example 79 Following the notation from Example 78, the center of Γ is

$$Z(\Gamma) = \{[0, 0, z] \colon z \in \mathbb{R}\},$$

hence $\Lambda' = \Lambda \cap Z(\Gamma)$ is a central subgroup of Γ contained in Λ and it is isomorphic to $\mathbb{Z}$. Fix a subgroup $\Delta \subseteq \Lambda'$. Then Λ/Δ is a discrete subgroup of Γ/Δ and the corresponding homogeneous space can be identified with Γ/Λ. Since $\mathrm{r}(\Lambda^w) = 2$ and $\mathrm{r}(\Delta) \leq 1$, we have $\mathrm{r}((\Lambda/\Delta)^w) \geq 1$, hence the homogeneous flow $\mathcal{F}_\Delta \colon \Gamma/\Delta \curvearrowright \Gamma/\Lambda$ has a free cycle by Proposition 77.

In the case when Λ is a normal subgroup of Γ, Propositions 80 and 82 apply.

Proposition 80 *Let Γ be a connected Lie group, $\Lambda \subseteq \Gamma$ be a uniform lattice and $\mathcal{F} \colon \Gamma \curvearrowright \Gamma/\Lambda$ be the homogeneous flow. If Λ is a normal subgroup of Γ then the following conditions are equivalent:*

(i) $\mathcal{F}$ has a free cycle,
(ii) Λ is infinite,
(iii) Γ is non-compact.

Proof Recall (say, from our proof of Theorem 75) that there is an epimorphism

$$\varrho \colon \pi_1(\Gamma/\Lambda) \to \Lambda.$$

Since Γ/Λ is a (compact) connected Lie group, its fundamental group $\pi_1(\Gamma/\Lambda)$ is finitely generated and abelian. Consequently, $\Lambda = \mathrm{im}(\varrho)$ is also finitely generated abelian, hence

$$\Lambda = \Lambda^{ab} = \Lambda^w \oplus \mathrm{tor}(\Lambda)$$

with Λ^w free abelian and $\mathrm{tor}(\Lambda)$, the torsion subgroup of Λ, finite. We infer that $\Lambda^w = 0$ if, and only if, Λ is finite. In view of Proposition 77, this verifies the equivalence of

(i) and (ii). The equivalence of (ii) and (iii) follows since $p\colon \Gamma \to \Gamma/\Lambda$ is a covering morphism with the compact base Γ/Λ and kernel Λ. □

By Corollary 76, conditions (ii) and (iii) from Proposition 80 are necessary for the validity of (i) also without the assumption of normality of Λ. However, they are in general not sufficient.

Example 81 Consider $\mathbb{T}^1$ as a subset of $\mathbb{C}$ with the multiplication as the group operation. By letting $\mathbb{T}^1$ act on $\mathbb{C}$ multiplicatively by automorphisms, we obtain the semidirect product $\Gamma = \mathbb{T}^1 \ltimes \mathbb{C}$ with the group operation

$$(z, x)(v, y) = (zv, v^{-1}x + y).$$

The subgroup $\Lambda = \mathbb{Z}_2 \ltimes \mathbb{Z}^2$ of Γ is discrete and cocompact, but not normal (the center of Γ being trivial). The homogeneous space Γ/Λ can be identified with the 3-dimensional analogue of the Klein bottle $\mathbb{T}^3/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting on $\mathbb{T}^3$ via the transformation

$$\varrho(z, v, w) = (-z, v^{-1}, w^{-1}).$$

Now, the commutator subgroup of Λ is $\Lambda' = 1 \times 2\mathbb{Z}^2$ and $\Lambda^{ab} = \Lambda/\Lambda'$ is isomorphic to $(\mathbb{Z}_2)^3$, hence is finite. Thus $\Lambda^w = 0$ and the homogeneous flow $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$ has no free cycles by Proposition 77.

Proposition 82 *Let Γ be a connected Lie group, $\Lambda \subseteq \Gamma$ be a uniform lattice and $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$ be the homogeneous flow. If Λ is a normal subgroup of Γ then*

$$\mathrm{r}(H_1^w(\mathcal{F})) = \mathrm{r}(H_1^w(\Gamma)).$$

In particular, $\mathcal{F}$ has a free cycle if, and only if,

$$\mathrm{r}(H_1^w(\Gamma)) < \mathrm{r}(H_1^w(\Gamma/\Lambda)). \tag{5.3}$$

Proof Recall the sequence

$$0 \longrightarrow \pi_1(\Gamma) \xrightarrow{p_*} \pi_1(\Gamma/\Lambda) \xrightarrow{r} \Lambda \longrightarrow 0 \tag{5.4}$$

from our proof of Theorem 75. As was mentioned therein, (5.4) is exact at $\pi_1(\Gamma/\Lambda)$ and Λ. The exactness at $\pi_1(\Gamma)$ follows from the fact that p is a covering map. Now both $\pi_1(\Gamma)$ and $\pi_1(\Gamma/\Lambda)$ are abelian finitely generated groups, hence so is Λ. It follows that (5.4) takes the form of the sequence

$$0 \longrightarrow H_1(\Gamma) \xrightarrow{p_*^{ab}} H_1(\Gamma/\Lambda) \xrightarrow{r^{ab}} \Lambda^{ab} \longrightarrow 0.$$

To prove the first claim of the proposition, we need only verify that p_*^w is a monomorphism. This follows since p_*^{ab}, being a monomorphism, restricts to a monomorphism between the torsion-free parts of $H_1(\Gamma)$ and $H_1(\Gamma/\Lambda)$, and this restriction coincides with p_*^w. The second claim of the proposition is now immediate. □

Condition (5.3) is sufficient for the existence of free cycles also in the case when Λ is not normal in Γ, the reason being an obvious inequality $\mathrm{r}(H_1^w(\mathcal{F})) \le \mathrm{r}(H_1^w(\Gamma))$. However, the necessity fails in this more general context.

Example 83 Fix a uniform lattice $\Upsilon \subseteq \mathrm{SL}(2,\mathbb{R})$ with $Y = \mathrm{SL}(2,\mathbb{R})/\Upsilon$ having a finite homology group $H_1(Y)$. (It seems to be well known that Brieskorn homology 3-spheres can be obtained in this way.) Let $\Gamma = \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}$, $\Lambda = \Upsilon \times \mathbb{Z}$ and consider the homogeneous flow $\mathcal{F}\colon \Gamma \curvearrowright \Gamma/\Lambda$. The projection $p\colon \Gamma \to \Gamma/\Lambda$ takes the form of the direct product of the projections $q\colon\ \mathrm{SL}(2,\mathbb{R}) \to Y$ and $r\colon \mathbb{R} \to \mathbb{T}^1$, hence p_*^w can be identified with the direct sum of $q_*^w : \mathbb{Z} \to 0$ and $r_*^w : 0 \to \mathbb{Z}$. Thus

$$\mathrm{r}(H_1^w(\Gamma)) = \mathrm{r}(H_1^w(\Gamma/\Lambda)) = 1$$

and

$$\mathrm{r}(H_1^w(\mathcal{F})) = \mathrm{r}(\mathrm{im}(p_*^w)) = 0.$$

It follows that $\mathcal{F}$ has a free cycle but (5.3) fails to hold.

5.4 Homogeneous flows induced by covering morphisms

Let Θ be a compact connected Lie group and $p_u\colon \Gamma_u \to \Theta$ be its universal cover. Then Γ_u carries the structure of a Lie group making p_u a Lie group morphism [20, Theorem 2.13, p. 43]. The subgroup $\Lambda_u = \ker(p_u)$ of Γ_u is normal and discrete, hence central by connectedness of Γ_u. Thus so is each subgroup Λ of Λ_u. Given a loop $f \in \pi_1(\Theta)$, consider the lift $\widetilde{f}$ of f across p_u starting at 1. The assignment $f \mapsto \widetilde{f}(1)^{-1}$ defines an isomorphism $r\colon \pi_1(\Theta) \to \Lambda_u$.

Fix a subgroup H of $\pi_1(\Theta)$ and set $\Lambda = r(H)$. Then p_u projects via the quotient morphism $\Gamma_u \to \Gamma_u/\Lambda$ onto a covering morphism $p\colon \Gamma_u/\Lambda \to \Theta$. We have $p_*(\pi_1(\Gamma_u/\Lambda)) = H$, hence, in view of [23, Theorem 79.2, p. 480], the assignment $H \mapsto p$ defines a bijection between the subgroups of $\pi_1(\Theta)$ and the equivalence classes of connected covers of Θ. Moreover, $\ker(p) = \Lambda_u/\Lambda \cong \pi_1(\Theta)/H$ and it follows that $\ker(p)$ is infinite if, and only if, so is $\pi_1(\Theta)/H$. In such a case we shall call H coinfinite in $\pi_1(\Theta)$.

Proposition 84 *Let Θ be a compact connected Lie group. Then the equivalence classes of connected covers of Θ inducing homogeneous flows on Θ with free cycles are in a one-to-one correspondence with the coinfinite subgroups of $\pi_1(\Theta)$.*

Proof In view of our discussion from the preceding paragraphs, it suffices to use the equivalence of (i) and (ii) from Proposition 80. □

Example 85 Given $n \in \mathbb{N}$, consider the unitary group $\mathrm{U}(n)$, its closed normal subgroup $\mathrm{SU}(n)$ and its 1-torus

$$T_1 = \{\mathrm{diag}(z, 1, \ldots, 1) \colon z \in \mathbb{T}^1\}.$$

Then $\mathrm{U}(n) = T_1 \cdot \mathrm{SU}(n)$ and $T_1 \cap \mathrm{SU}(n) = \{1\}$, hence $\mathrm{U}(n) = T_1 \ltimes \mathrm{SU}(n)$. Since $\mathrm{SU}(n)$ is simply connected, the universal cover of $\mathrm{U}(n)$ is a semidirect product $\mathbb{R} \ltimes \mathrm{SU}(n)$ and the kernel of the corresponding covering morphism p_u is $\mathbb{Z} \times \{1\}$. Thus, since $\pi_1(\mathrm{U}(n))$ is infinite cyclic, the only coinfinite subgroup of $\pi_1(\mathrm{U}(n))$ is the trivial one. Consequently, by Proposition 84, the only connected cover of $\mathrm{U}(n)$ inducing a homogeneous flow on $\mathrm{U}(n)$ with a free cycle is p_u.

Example 86 Given $n \geq 2$, consider $\mathrm{SO}(n)$ as a closed subgroup of $\mathrm{U}(n)$ and let it act on $\mathrm{U}(n)$ by inner automorphisms. The corresponding semidirect product satisfies

$$\pi_1(\mathrm{SO}(n) \ltimes \mathrm{U}(n)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}, & \text{if } n \geq 3, \end{cases}$$

[10, p. 335]. Thus, if $n \geq 3$ then $\pi_1(\mathrm{SO}(n) \ltimes \mathrm{U}(n))$ has exactly two coinfinite subgroups, namely 0 and $\mathbb{Z}_2$. As for $n = 2$, there are infinitely many coinfinite subgroups of $\pi_1(\mathrm{SO}(2) \ltimes \mathrm{U}(2))$, namely the trivial one and those of rank 1.

5.5 Actions of PSL(2, $\mathbb{R}$)

In this subsection we shall work with Fuchsian groups, that is, the discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$. As is customary in the literature, we shall denote such a group by Γ, thus temporarily abandoning the notation used so far. Concerning basic facts on Fuchsian groups used below, we refer to [18].

The group $\mathrm{PSL}(2, \mathbb{R})$ acts effectively, isometrically and transitively on the hyperbolic plane $\mathbf{H}$ via Möbius transformations. A subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ is Fuchsian if, and only if, it acts on $\mathbf{H}$ properly discontinuously. The action need not be free, its freeness being equivalent to the absence of elliptic elements in Γ. If that is the case then the projection $p \colon \mathbf{H} \to \mathbf{H}/\Gamma$ is a covering map and $\mathbf{H}/\Gamma$ is a Riemann surface with $\pi_1(\mathbf{H}/\Gamma)$ isomorphic to Γ. We write $X_\Gamma = \mathrm{PSL}(2, \mathbb{R})/\Gamma$ and consider the homogeneous action $\mathrm{PSL}(2, \mathbb{R}) \curvearrowright X_\Gamma$. Recall that X_Γ is compact if, and only if, so is $\mathbf{H}/\Gamma$, in which case Γ is nonabelian.

Proposition 87 *Let* $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ *be a cocompact Fuchsian group without elliptic elements and* $\mathcal{F}_\Gamma \colon \mathrm{PSL}(2, \mathbb{R}) \curvearrowright X_\Gamma$ *be the homogeneous flow. Then*

$$\mathrm{r}(H_1^w(X_\Gamma)) - \mathrm{r}(H_1^w(\mathcal{F}_\Gamma)) = \mathrm{r}(H_1(\mathbf{H}/\Gamma)) \geq 4, \tag{5.5}$$

hence $\mathcal{F}_\Gamma$ *has a free cycle.*

Proof Let g denote the genus of $\mathbf{H}/\Gamma$. By compactness of X_Γ, Γ is nonabelian and since $\Gamma \cong \pi_1(\mathbf{H}/\Gamma)$, we infer that $g \geq 2$. Further,

$$\Gamma^{ab} \cong H_1(\mathbf{H}/\Gamma) \cong \mathbb{Z}^{2g},$$

hence

$$\mathrm{r}(\Gamma^w) = \mathrm{r}(H_1(\mathbf{H}/\Gamma)) = 2g \geq 4.$$

Now it suffices to apply Theorem 75. □

The assumption of Γ containing no elliptic elements can not be omitted from Proposition 87.

Example 88 Similarly to Example 83, choose a cocompact Fuchsian group Γ with $H_1(X_\Gamma)$ finite. Since $\pi_1(X_\Gamma)$ factors onto Γ, $H_1(X_\Gamma)$ factors onto Γ^{ab}, hence Γ^{ab} is finite and $\Gamma^w = 0$. Thus it follows from Proposition 77 that the associated flow $\mathcal{F}_\Gamma$ has no free cycle.

Remark 89 Recall that every compact Riemann surface Y with genus $g \geq 2$ is conformally equivalent to $\mathbf{H}/\Gamma$ for some freely acting Fuchsian group Γ [24, Theorem 5.2, p. 48]. Since $\mathrm{r}(H_1(Y)) = 2g$, it follows from (5.5) that g is uniquely determined by the isomorphism class of $\mathcal{F}_\Gamma$. Thus we have infinitely many mutually nonisomorphic flows $\mathcal{F}_\Gamma$ with free cycles by Proposition 87. In fact, Remark 91 claims more than that.

For the purpose of the following proposition we recall some basic facts concerning equivariant maps between homogeneous spaces. Let G be a group with subgroups Γ and Γ'. Consider the projections $p\colon G \to G/\Gamma$, $p'\colon G \to G/\Gamma'$ and the homogeneous actions $\mathcal{F}\colon G \curvearrowright G/\Gamma$, $\mathcal{F}'\colon G \curvearrowright G/\Gamma'$. A map $h\colon G/\Gamma \to G/\Gamma'$ is G-equivariant if, and only if, it lifts across p and p' to a right translation of G. Further, given $g \in G$, the right translation of G by g projects via p and p' to a G-equivariant map if, and only if, $\Gamma g \subseteq g\Gamma'$. It follows that a G-equivariant bijection $h\colon G/\Gamma \to G/\Gamma'$ exists if, and only if, Γ, Γ' are conjugate in G.

Now let, in addition, G be a connected Lie group and Γ, Γ' be discrete cocompact subgroups of G. Then all translations of G are diffeomorphisms and both p, p' are smooth covering maps. Consequently, every G-equivariant map h is smooth and every G-equivariant bijection is a diffeomorphism. Thus the flows $\mathcal{F}, \mathcal{F}'$ are isomorphic if, and only if, Γ, Γ' are conjugate in G.

Proposition 90 *Let* $\Gamma, \Gamma' \subseteq \mathrm{PSL}(2, \mathbb{R})$ *be cocompact Fuchsian groups without elliptic elements. Then the following conditions are equivalent:*

(1) the flows $\mathcal{F}_\Gamma, \mathcal{F}_{\Gamma'}$ *are isomorphic,*
(2) the groups Γ, Γ' *are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$,
(3) the Riemann surfaces $\mathbf{H}/\Gamma, \mathbf{H}/\Gamma'$ *are conformally equivalent.*

Remark 91 It follows from the proposition that for every integer $g \geq 2$, the isomorphism classes of flows $\mathcal{F}_\Gamma$ with $\mathrm{r}(\Gamma^{ab}) = 2g$ are in a one-to-one correspondence with the elements of the moduli space of conformal structures on the compact surface of genus g.

Proof The equivalence of (1) and (2) follows from our discussion preceding the proposition. The equivalence of (2) and (3) is well known. □

5.6 Changing the acting group

The purpose of this subsection is to investigate how the existence of free cycles is affected by changing the acting group of a given minimal flow. Our main results here are Propositions 93 and 94, where we discuss passing to quotient groups, extensions and subgroups. We begin with a simple algebraic lemma.

Lemma 92 *Let A, B be free abelian groups with finite ranks and C, D be their subgroups in the respective order. Let an epimorphism $f\colon A \to B$ restrict to an epimorphism between C and D. If $\mathrm{r}(C) = \mathrm{r}(A)$ then $\mathrm{r}(D) = \mathrm{r}(B)$.*

Proof By assumptions, f descends to an epimorphism between A/C and B/D. Since $\mathrm{r}(C) = \mathrm{r}(A)$, A/C is a torsion group, hence so is B/D. Thus $\mathrm{r}(D) = \mathrm{r}(B)$. □

Given a flow $\mathcal{F}\colon \Gamma \curvearrowright X$, a connected Lie group Γ' and a quotient morphism $q\colon \Gamma' \to \Gamma$, we shall consider the flow $\mathcal{F}'\colon \Gamma' \curvearrowright X$ given by $\mathcal{F}'(\alpha, x) = \mathcal{F}(q(\alpha), x)$ for $\alpha \in \Gamma'$ and $x \in X$. Clearly, $\mathcal{F}$ and $\mathcal{F}'$ share orbits, hence minimality of one of them is equivalent to that of the other.

Proposition 93 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow, Γ' be a connected Lie group and $q\colon \Gamma' \to \Gamma$ be a quotient morphism with the finite kernel. Then $\mathrm{r}(H_1^w(\mathcal{F}')) = \mathrm{r}(H_1^w(\mathcal{F}))$. Consequently, $\mathcal{F}'$ has a free cycle if, and only if, so does $\mathcal{F}$.*

Proof Fix $z \in X$. We shall prove the proposition by applying Lemma 92 to the epimorphism

$$(\mathcal{F}_z)_*^w\colon H_1^w(\Gamma) \to H_1^w(\mathcal{F})$$

and subgroups

$$\mathrm{im}(q_*^w) \subseteq H_1^w(\Gamma) \quad \text{and} \quad H_1^w(\mathcal{F}') \subseteq H_1^w(\mathcal{F}).$$

In the following two steps we verify that the assumptions of Lemma 92 are fulfilled.

Step 1. We show that $(\mathcal{F}_z)_*^w$ restricts to an epimorphism between $\mathrm{im}(q_*^w)$ and $H_1^w(\mathcal{F}')$.

By definition of $\mathcal{F}'$, $\mathcal{F}'_z = \mathcal{F}_z q$, hence $(\mathcal{F}'_z)_*^w = (\mathcal{F}_z)_*^w q_*^w$ and it follows that

$$(\mathcal{F}_z)_*^w(\mathrm{im}(q_*^w)) = \mathrm{im}((\mathcal{F}'_z)_*^w) = H_1^w(\mathcal{F}').$$

Step 2. We show that $H_1^w(\Gamma)$ and $\mathrm{im}(q_*^w)$ have the same rank.

Fig. 5 The commutative diagram defining q_*^w

$$\begin{tikzcd}
H_1(\Gamma') \arrow[r, "q_*^{ab}"] \arrow[d, "p'"] & H_1(\Gamma) \arrow[d, "p"] \\
H_1^w(\Gamma') \arrow[r, "q_*^w"] & H_1^w(\Gamma)
\end{tikzcd}$$

The morphism q induces a short exact sequence

$$0 \longrightarrow \pi_1(\Gamma') \xrightarrow{q_*} \pi_1(\Gamma) \longrightarrow \ker(q) \longrightarrow 0. \tag{5.6}$$

Since all the groups in (5.6) are abelian, the sequence takes the form

$$0 \longrightarrow H_1(\Gamma') \xrightarrow{q_*^{ab}} H_1(\Gamma) \longrightarrow \ker(q) \longrightarrow 0.$$

Since $\ker(q)$ is finite by assumption and q_*^{ab} is a monomorphism,

$$\mathrm{r}(H_1^w(\Gamma)) = \mathrm{r}(H_1(\Gamma)) = \mathrm{r}(H_1(\Gamma')) = \mathrm{r}(\mathrm{im}(q_*^{ab})).$$

Now, consider the commutative diagram in Fig. 5 with p', p being the canonical projections.

Clearly, $\mathrm{im}(q_*^w) = p(\mathrm{im}(q_*^{ab}))$. Since $\ker(p) = \mathrm{tor}(H_1(\Gamma))$ is a torsion group, p preserves ranks of subgroups, hence $\mathrm{r}(\mathrm{im}(q_*^w)) = \mathrm{r}(\mathrm{im}(q_*^{ab}))$.

In summary,

$$\mathrm{r}(H_1^w(\Gamma)) = \mathrm{r}(\mathrm{im}(q_*^{ab})) = \mathrm{r}(\mathrm{im}(q_*^w)),$$

as was to be shown. □

We postpone examples related to Proposition 93 to Sect. 5.7.

Proposition 94 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow and Υ be a closed connected subgroup of Γ. Assume that the flow $\widetilde{\mathcal{F}}\colon \Upsilon \curvearrowright X$ obtained by restricting $\mathcal{F}$ is minimal. Then $\mathrm{r}(H_1^w(\widetilde{\mathcal{F}})) \le \mathrm{r}(H_1^w(\mathcal{F}))$. In particular, if $\mathcal{F}$ has a free cycle then so does $\widetilde{\mathcal{F}}$.*

Proof Fix $z \in X$ and consider the inclusion morphism $j\colon \Upsilon \to \Gamma$. Then $\widetilde{\mathcal{F}}_z = \mathcal{F}_z j$, hence $(\widetilde{\mathcal{F}}_z)_*^w = (\mathcal{F}_z)_*^w j_*^w$ and we infer that

$$H_1^w(\widetilde{\mathcal{F}}) = \mathrm{im}((\widetilde{\mathcal{F}}_z)_*^w) = (\mathcal{F}_z)_*^w(\mathrm{im}(j_*^w)) \subseteq \mathrm{im}((\mathcal{F}_z)_*^w) = H_1^w(\mathcal{F}).$$

The desired inequality now follows. □

Contrary to Proposition 93, an equivalence can not be claimed in Proposition 94.

Example 95 Let $\Gamma = \mathbb{R} \times \mathbb{T}^1$, $X = \mathbb{T}^1$ and consider the flow $\mathcal{F}\colon \Gamma \curvearrowright X$ acting by $\mathcal{F}(t, x, y) = e^{i2\pi t}xy$. Clearly, $\mathcal{F}$ may be identified with the homogeneous flow $\Gamma \curvearrowright \Gamma/\Lambda$, where

$$\Lambda = \{(t, e^{-i2\pi t})\colon t \in \mathbb{R}\}.$$

Choose $z = 1$ as the base point of X. Then the transition map $\mathcal{F}_z\colon \Gamma \to X$ acts by the rule $\mathcal{F}_z(t, x) = e^{i2\pi t}x$, inducing thus an isomorphism $(\mathcal{F}_z)^w_*\colon H^w_1(\Gamma) \to H^w_1(X)$. It follows that the flow $\mathcal{F}$ has no free cycle.

Now let $\Upsilon = \mathbb{R} \times \{1\}$ and consider the restricted flow $\widetilde{\mathcal{F}}\colon \Upsilon \curvearrowright X$. Then $\widetilde{\mathcal{F}}$ is transitive, hence minimal, and since Υ is simply connected,

$$\mathrm{r}(H^w_1(\widetilde{\mathcal{F}})) = \mathrm{r}(H^w_1(\Upsilon)) = 0 < 1 = \mathrm{r}(H^w_1(X)).$$

Thus, unlike $\mathcal{F}$, $\widetilde{\mathcal{F}}$ does have a free cycle.

Remark 96 While the existence of free cycles gets inherited by restricted flows, minimality does not in general and it is an important (and difficult) problem in homogeneous dynamics to determine when the restricted flow inherits minimality. Let us mention here at least two of the most classical results in this direction, namely Hedlund's theorem on minimality of horocycle flows [13] and a theorem of Auslander, Green and Hahn on minimality of homogeneous flows on nilmanifolds [3].

5.7 Actions of SL(2, $\mathbb{R}$)

In this subsection we describe examples of homogeneous flows with free cycles having $\mathrm{SL}(2, \mathbb{R})$ as their acting group. We use basic tools from the theory of arithmetic Fuchsian groups, referring to [18, Chapter 5].

Let $\mathrm{M}_2(\mathbb{R})$ denote the algebra of the real 2×2 matrices. Given positive integers a, b, we shall consider the basis of $\mathrm{M}_2(\mathbb{R})$ formed by the matrices

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & 0 \end{bmatrix}.$$

With the notation $x = x1$ for $x \in \mathbb{R}$ we have

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

Moreover, for all $x, y, z, v \in \mathbb{R}$,

$$x + yi + zj + vk = \begin{bmatrix} x + \sqrt{a}y & z + \sqrt{a}v \\ b(z - \sqrt{a}v) & x - \sqrt{a}y \end{bmatrix},$$

hence

$$\det(x + yi + zj + vk) = x^2 - ay^2 - bz^2 + abv^2 \quad \text{and} \quad \mathrm{tr}(x + yi + zj + vk) = 2x.$$

Thus

$$\mathrm{SL}(2,\mathbb{R}) = \{x + yi + zj + vk \colon x, y, z, v \in \mathbb{R}, x^2 - ay^2 - bz^2 + abv^2 = 1\}$$

and

$$\Upsilon = \{x + yi + zj + vk \colon x, y, z, v \in \mathbb{Z}, x^2 - ay^2 - bz^2 + abv^2 = 1\}$$

is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. We recall from [18, Theorems 5.2.5, 5.4.1] that Υ is cocompact in $\mathrm{SL}(2,\mathbb{R})$ when b is a prime number and a is a quadratic nonresidue modulo b.

The following lemma is probably well known. Since we are unable to give a reference, we include a proof, following the line of that of [18, Theorem 5.2.5, p. 116].

Lemma 97 *Let b be a prime number of the form $b = c^2 + 1$ with $c \geq 2$ an integer and let $a \geq 2$ be a quadratic nonresidue modulo b. Then Υ contains no elliptic matrices.*

Remark 98 It is a longstanding open problem of number theory, known as the fourth Landau's problem, as to whether the prime numbers of the form $b = c^2 + 1$ form an infinite set. On the other hand, given any prime number $b \geq 3$, there do exist quadratic nonresidues $a \geq 2$ modulo b, for the multiplicative group $\mathbb{Z}_b^\times$ of the filed $\mathbb{Z}_b$ is cyclic with even order $b - 1$, hence the quadratic residues from $\mathbb{Z}_b^\times$ form a subgroup of $\mathbb{Z}_b^\times$ of index 2.

Proof We proceed by contradiction. Fix $x, y, z, v \in \mathbb{Z}$ with $x^2 - ay^2 - bz^2 + abv^2 = 1$ and let the matrix $x + yi + zj + vk \in \Upsilon$ be elliptic, that is, $|\operatorname{tr}(x + yi + zj + vk)| < 2$. Then $x = 0$ and $-ay^2 - bz^2 + abv^2 = 1$. Thus b divides $ay^2 + 1$, hence it does not divide y. Consequently, y has a multiplicative inverse w modulo b, that is, $yw \equiv_b 1$. Then $ay^2 \equiv_b -1 \equiv_b c^2$, hence $a \equiv_b (cw)^2$, contradicting our assumptions on a. □

Consider the usual 2-to-1 quotient morphism $p\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$, the Fuchsian group $\Gamma = p(\Upsilon)$ and the homogeneous flow $\mathcal{F}_\Gamma\colon \mathrm{PSL}(2,\mathbb{R}) \curvearrowright X_\Gamma$ from Sect. 5.5. Let $\mathcal{F}'_\Gamma\colon \mathrm{SL}(2,\mathbb{R}) \curvearrowright X_\Gamma$ be the flow induced by $\mathcal{F}_\Gamma$ and p, as explained in Sect. 5.6. Since Υ is p-saturated, X_Γ may be identified with $Y = \mathrm{SL}(2,\mathbb{R})/\Upsilon$ and $\mathcal{F}'_\Gamma$ is isomorphic to the homogeneous flow $\mathcal{F}'\colon \mathrm{SL}(2,\mathbb{R}) \curvearrowright Y$.

Proposition 99 *Let b be a prime number of the form $b = c^2 + 1$ with $c \geq 2$ an integer and let $a \geq 2$ be a quadratic nonresidue modulo b. Then*

$$\mathrm{r}(H_1^w(Y)) - \mathrm{r}(H_1^w(\mathcal{F}')) = \mathrm{r}(H_1(\mathbf{H}/\Gamma)) \geq 4,$$

hence $\mathcal{F}'$ has a free cycle.

Proof As explained above, we may consider $\mathcal{F}'_\Gamma$ instead of $\mathcal{F}'$. By Lemma 97, Γ fits into the setting of our Sect. 5.5. Thus it follows from Propositions 93 and 87 that

$$\mathrm{r}(H_1^w(X_\Gamma)) - \mathrm{r}(H_1^w(\mathcal{F}'_\Gamma)) = \mathrm{r}(H_1^w(X_\Gamma)) - \mathrm{r}(H_1^w(\mathcal{F}_\Gamma)) = \mathrm{r}(H_1(\mathbf{H}/\Gamma)) \geq 4.$$

□

Remark 100 One can proceed also in the opposite direction, starting by fixing a cocompact freely acting Fuchsian group Γ and then considering the flow $\mathcal{F}'_\Gamma$ induced by $\mathcal{F}_\Gamma$ and p. Similarly to Proposition 99, $\mathcal{F}'_\Gamma$ has a free cycle on the account of Propositions 93 and 87.

5.8 Changing the phase space

Our aim in this subsection is to study the inheritance of free cycles by extensions and factors. Two sufficient conditions are given in Proposition 101.

Let $\mathcal{F}\colon \Gamma \curvearrowright X, \mathcal{G}\colon \Gamma \curvearrowright Y$ be minimal flows. A morphism $\mathcal{F} \to \mathcal{G}$ is a continuous equivariant map $p\colon X \to Y$. By minimality, every morphism $\mathcal{F} \to \mathcal{G}$ is surjective, hence is a factor map.

Proposition 101 *Let $\mathcal{F}\colon \Gamma \curvearrowright X$ and $\mathcal{G}\colon \Gamma \curvearrowright Y$ be minimal flows, $p\colon \mathcal{F} \to \mathcal{G}$ be a morphism and $z \in X$. Then*

$$p_*^w(\mathcal{F}_z)_*^w = (\mathcal{G}_{p(z)})_*^w \quad \text{and} \quad H_1^w(\mathcal{G}) = p_*^w(H_1^w(\mathcal{F})). \tag{5.7}$$

Moreover, the following statements hold:

(1) if $\mathcal{F}$ has a free cycle and p_*^w is a monomorphism then $\mathcal{G}$ has a free cycle,
(2) if $\mathcal{G}$ has a free cycle and p_*^w is an epimorphism then $\mathcal{F}$ has a free cycle.

Proof We divide the proof into three steps.

STEP 1. We verify (5.7).

Since p is a morphism, we have $p\mathcal{F}_z = \mathcal{G}_{p(z)}$, hence $p_*^w(\mathcal{F}_z)_*^w = (\mathcal{G}_{p(z)})_*^w$. Thus

$$H_1^w(\mathcal{G}) = \mathrm{im}((\mathcal{G}_{p(z)})_*^w) = p_*^w(\mathrm{im}((\mathcal{F}_z)_*^w)) = p_*^w(H_1^w(\mathcal{F})).$$

STEP 2. We verify (1).

If p_*^w is a monomorphism then $\mathrm{r}(H_1^w(X)) \leq \mathrm{r}(H_1^w(Y))$ and

$$\mathrm{r}(H_1^w(\mathcal{G})) = \mathrm{r}(p_*^w(H_1^w(\mathcal{F}))) = \mathrm{r}(H_1^w(\mathcal{F})).$$

Assuming that $\mathcal{F}$ has a free cycle, we infer that

$$\mathrm{r}(H_1^w(\mathcal{G})) = \mathrm{r}(H_1^w(\mathcal{F})) < \mathrm{r}(H_1^w(X)) \leq \mathrm{r}(H_1^w(Y)).$$

STEP 3. We verify (2).

Let $\mathcal{G}$ have a free cycle and p_*^w be an epimorphism. By (5.7), the restriction

$$q = p_*^w|_{H_1^w(\mathcal{F})}\colon H_1^w(\mathcal{F}) \to H_1^w(\mathcal{G})$$

is also an epimorphism and $\ker(q) \subseteq \ker(p_*^w)$. Consequently,

$$\begin{aligned}\mathrm{r}(H_1^w(\mathcal{F})) &= \mathrm{r}(H_1^w(\mathcal{G})) + \mathrm{r}(\ker(q)) < \mathrm{r}(H_1^w(Y)) + \mathrm{r}(\ker(q)) \\ &= \mathrm{r}(H_1^w(X)) - \mathrm{r}(\ker(p_*^w)) + \mathrm{r}(\ker(q)) \leq \mathrm{r}(H_1^w(X)).\end{aligned}$$

□

As an application of Proposition 101, let us discuss free cycles in group extensions of minimal flows.

Example 102 Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow. Assume that the manifold X is smooth and let a compact connected Lie group G act on X smoothly, freely and equivariantly with respect to $\mathcal{F}$. Then X/G carries the structure of a (compact connected) smooth manifold making the projection $p\colon X \to X/G$ a submersion. By equivariance, $\mathcal{F}$ factors via p onto a minimal flow $\mathcal{G}\colon \Gamma \curvearrowright X/G$. Since p is a submersion, it has local cross sections, hence is a bundle projection with the fibre G. By connectedness of G, it follows that $p_*\colon \pi_1(X) \to \pi_1(X/G)$ is an epimorphism, hence so is $p_*^w\colon H_1^w(X) \to H_1^w(X/G)$. In view of Proposition 101(2) this means that if $\mathcal{G}$ has a free cycle then so does $\mathcal{F}$.

Example 103 Consider the situation described in Example 102, but assume, in place of connectedness, that G is finite. Then the projection $p\colon X \to X/G$ is a covering map, hence $p_*\colon \pi_1(X) \to \pi_1(X/G)$ is a monomorphism. If $\pi_1(X/G)$ is abelian then so is $\pi_1(X)$ and $p_* = p_*^{ab}$. Consequently, p_*^w is a monomorphism and Proposition 101(1) applies.

Existence of free cycles is carried over also to skew product extensions.

Example 104 Let $\mathcal{F}\colon \Gamma \curvearrowright X$ and $\mathcal{G}\colon \Gamma \curvearrowright Y$ be minimal flows, $\mathcal{F}$ being a skew product over $\mathcal{G}$ with $X = Y \times Z$. Then the projection $p\colon X \to Y$ is a morphism of flows $\mathcal{F} \to \mathcal{G}$. Under the usual identification $H_1^w(X) = H_1^w(Y) \times H_1^w(Z)$, the induced morphism p_*^w takes the form of the projection $H_1^w(Y) \times H_1^w(Z) \to H_1^w(Y)$, hence it is an epimorphism. Thus it follows from Proposition 101(2) that $\mathcal{F}$ has a free cycle if $\mathcal{G}$ has a free cycle. If, in addition, $H_1^w(Z) = 0$ (which is the case, in particular, if Z is simply connected) then p_*^w is an isomorphism. In view of Proposition 101 this implies that $\mathcal{F}$ has a free cycle if, and only if, so does $\mathcal{G}$.

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References

1. Anosov, D.V., Katok, A.B.: New examples in smooth ergodic theory. Ergodic diffeomorphisms (Russian). Trudy Moskov. Mat. Obšč. **23**, 3–36 (1970)
2. Arvanitoyeorgos, A.: An introduction to Lie groups and the geometry of homogeneous spaces. Translated from the 1999 Greek original and revised by the author. Student Mathematical Library, 22. American Mathematical Society, Providence, RI, xvi+141 (2003)

3. Auslander, L., Green, L., Hahn, F.: Flows on some three-dimensional homogeneous spaces. Bull. Am. Math. Soc. **67**, 494–497 (1961)
4. Dirbák, M.: Minimal extensions of flows with amenable acting groups. Israel J. Math. **207**(2), 581–615 (2015)
5. Dirbák, M.: First cohomology groups of minimal flows. Dissertationes Math. **562**, 206 (2021)
6. Dirbák, M., Snoha, L'., Špitalský, V.: Minimal direct products. Trans. Am. Math. Soc. **375**(9), 6453–6506 (2022)
7. Fathi, A., Herman, M.R.: Existence de difféomorphismes minimaux. (French) Dynamical systems, Vol. I-Warsaw, pp. 37–59. Astérisque, No. 49, Soc. Math. France, Paris (1977)
8. Fayad, B., Katok, A.: Constructions in elliptic dynamics. Ergodic Theory Dynam. Syst. **24**(5), 1477–1520 (2004)
9. Glasner, S., Weiss, B.: On the construction of minimal skew products. Israel J. Math. **34**(1979), 4, 321–336 (1980)
10. Hall, B.C.: Lie groups, Lie algebras, and representations. An elementary introduction. Graduate Texts in Mathematics, Vol. 222. Springer-Verlag, New York (2003)
11. Hasselblatt, B.: Anatole Katok. Ergodic Theory Dynam. Syst. **42**(2), 321–388 (2022)
12. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
13. Hedlund, G.A.: Fuchsian groups and transitive horocycles. Duke Math. J. **2**(3), 530–542 (1936)
14. Hirsch, M.W.: Differential Topology. Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg (1976)
15. Hofmann, K.H., Morris, S.A.: The structure of compact groups. A primer for the student-a handbook for the expert. Second revised and augmented edition. De Gruyter Studies in Mathematics, Vol. 25. Walter de Gruyter & Co., Berlin (2006)
16. Illanes, A., Nadler, S.B., Jr.: Hyperspaces. Fundamentals and recent advances. In: Monographs and Textbooks in Pure and Applied Mathematics, Vol. 216. Marcel Dekker, Inc., New York (1999)
17. Katok, A.B.: Minimal diffeomorphisms on principal S^1-bundles (Russian), abstracts of 6 All-Union Topological Conference, Tbilisi, p. 63 (1972)
18. Katok, S.: Fuchsian Groups. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL (1992)
19. Kobayashi, S.: Transformation groups in differential geometry. Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin (1995)
20. Lee, J.M.: Introduction to Smooth Manifolds. Graduate Texts in Mathematics, Vol. 218. Springer-Verlag, New York (2003)
21. Lima, E.L.: Fundamental Groups and Covering Spaces. Translated from the Portuguese by Jonas Gomes. A K Peters, Ltd., Natick, MA (2003)
22. Munkres, J.R.: Elements of Algebraic Topology. Addison-Wesley Publishing Company, Menlo Park, CA (1984)
23. Munkres, J.R.: Topology. Second Edition of [MR0464128]. Prentice Hall, Inc., Upper Saddle River, NJ (2000)
24. Natanzon, S.M.: Complex Analysis, Riemann Surfaces and Integrable Systems. Moscow Lectures, Vol. 3, p. 139. Springer, Cham (2019)
25. Nerurkar, M.G.: On the construction of smooth ergodic skew-products. Ergodic Theory Dynam. Syst. **8**(2), 311–326 (1988)
26. Tapp, K.: Matrix Groups for Undergraduates. Student Mathematical Library, 29. American Mathematical Society, Providence, RI (2005)